MATHEMATICAL THEORY OF FEYNMAN PATH INTEGRALS *

by

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A B S T R A C T

We develop a general theory of oscillating integrals on real Hilbert space and we apply it to the mathematical foundation of the so called Feynman path integrals of non relativistic quantum mechanics, quantum statistical mechanics and quantum field theory.

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Contents

1. Introduction

Feyman path integrals have been introduced by Feynman in his formulation of quantum mechanics [1].[1)] Since their inception they have occupied a somewhat ambiguous position in theoretical physics. On one hand they have been widely and profitably used in quantum mechanics, statistical mechanics and quantum field theory, because of their strong intuitive, heuristic and formal appeal. On the other hand most of their uses have not been supported by an adequate mathematical justification. Especially in view of the potentialities of Feynman's approach as an alternative formulation of quantum dynamics, the need for a mathematical foundation has been broadly felt and the mathematical study of Feynman path integrals repeatedly strongly advocated, see e.g. [4] . This is roughly speaking, a study of oscillating integrals in infinitely many dimensions, hence closely connected with the development of the theory of integration in function spaces, see e.g. [5] . The present work intends to give a mathematical theory of Feynman path integrals and to yield applications to non relativistic quantum mechanics, statistical mechanics and quantum field theory. In order to establish connections with previous work, we shall give in this introduction a short historical sketch of the mathematical foundations of Feynman path integrals. For more details we refer to the references, in particular to the review papers [6].

Let us first briefly sketch the heuristic idea of Feynman path integrals, considering the simple case of a non relativistic particle of mass m , moving in Euclidean space R^n under the influence of a conservative force given by the potential $V(x)$, which we assume, for simplicity, to be a bounded continuous real valued function on R^n .

The classical Lagrangian, from which the classical Euler-Lagrange equations of motion follow, is

$$L(x, \frac{dx}{dt}) = \frac{1}{2m}(\frac{dx}{dt})^2 - V(x) . \tag{1.1}$$

Hamilton's principle of least action states that the trajectory actually followed by the particle going from the point y , at time zero, to the point x at time t , is the one which makes the classical action, i.e. Hamilton's principal function,

$$S_t(\gamma) = \int_0^t L(\gamma(\tau), \frac{d\gamma(\tau)}{d\tau})d\tau \tag{1.2}$$

stationary, under variations of the path $\gamma = \{\gamma(\tau)\}$, $0 \le \tau \le t$, with $\gamma(0) = y$ and $\gamma(t) = x$, which leave fixed the initial and end points y and x .

In quantum mechanics the state of the particle at time t is described by a function $\psi(x,t)$ which, for every t , belongs to $L_2(R^n)$ and satisfies Schrödinger's equation of motion

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = - \frac{\hbar^2}{2m} \Delta\psi(x,t) + V(x)\psi(x,t) , \tag{1.3}$$

with prescribed Cauchy data at time $t = 0$,

$$\psi(x,0) = \varphi(x) , \tag{1.4}$$

where Δ is the Laplacian on R^n and $\hbar$ is Planck's constant divided by 2π . The operator

$$H = - \frac{\hbar^2}{2m} \Delta + V(x) , \tag{1.5}$$

the Hamiltonian of the quantum mechanical particle, is self-adjoint on the natural domain of Δ and therefore

$e^{-\frac{i}{\hbar}tH}$ is a strongly continuous unitary group on $L_2(R^n)$. The solution of the initial value problem (1.3), (1.4) is

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$$\psi(x,t) = e^{-\frac{i}{\hbar}tH}\varphi(x) . \tag{1.6}$$

From the Lie-Kato-Trotter product formula we have

$$e^{-\frac{i}{\hbar}tH} = s-\lim_{k\to\infty} \left(e^{-\frac{i}{\hbar}\frac{t}{k}H_o}\, e^{-\frac{i}{\hbar}\frac{t}{k}V}\right)^k , \tag{1.7}$$

where

$$H_o = -\frac{\hbar^2}{2m}\Delta . \tag{1.8}$$

Assuming now for simplicity that φ is taken in Schwartz space $\mathcal{S}(R^n)$, we have, on the other hand

$$e^{-\frac{i}{\hbar}tH_o}\varphi(x) = (2\pi i\frac{\hbar}{m}t)^{-n/2}\int e^{im(x-y)^2/2\hbar t}\varphi(y)dy , \tag{1.9}$$

hence, combining (1.7) and (1.9)

$$e^{-\frac{i}{\hbar}tH}\varphi(x) = s-\lim_{k\to\infty}(2\pi i\frac{\hbar}{m}t)^{-kn/2}\int_{R^n} e^{\frac{i}{\hbar}S_t(x_k,\ldots,x_o)}\varphi(x_o)\, dx_o\ldots dx_{k-1}) \tag{1.10}$$

where by definition $x_k = x$ and

$$S_t(x_k,\ldots,x_o) = \sum_{j=1}^{k}\left[\frac{m}{2}\frac{(x_j-x_{j-1})^2}{(t/k)^2} - V(x_j)\right]\frac{t}{k} . \tag{1.11}$$

The expression (1.10) gives the solution of Schrödinger's equation as a limit of integrals.

Feynman's idea can now be formulated as the attempt to rewrite (1.10) in such a may that it appears, formally at least, as an integral over a space of continuous functions, called paths. Let namely $\gamma(\tau)$ be a real absolutely continuous function on the interval $[0,t]$, such that $\gamma(\tau_j) = x_j$, $j = 0,\ldots,k$, where $\tau_j = jt/k$ and $x_0,\ldots,x_k$ are given points in R^n, with $x_k = x$. Feynman looks upon $S_t(x_k,\ldots,x_0)$ as a Riemann appoximation for the classical action $S_t(\gamma)$ along the path γ,

$$S_t(\gamma) = \int_0^t \frac{m}{2} \left(\frac{d\gamma}{d\tau}\right)^2 d\tau - \int_0^t V(\gamma(\tau))d\tau . \tag{1.12}$$

Moreover when $k \to \infty$ the measure in (1.10) becomes formally $d\gamma = \prod_{0\leq\tau\leq t} \left(2\pi i \frac{\hbar}{m}\right)^{-n/2} d\gamma(\tau)$, so that (1.10) becomes the heuristic expression

$$\int_{\gamma(t)=x} e^{\frac{i}{\hbar}S_t(\gamma)} \varphi(\gamma(0))d\gamma , \tag{1.13}$$

where the integration should be over a suitable set of paths ending at time t at the point x. This is Feynman's path integral expression for the solution of Schrödinger's equation and we shall now review some of the work that has been done on its mathematical foundation.[2] Integration theory in spaces of continuous functions was actually available well before the advent of Feynman path integrals, particularly originated by Wiener's work (1921) on the Brownian motion, see e.g. [7]. It was however under the influence of Feynman's work that Kac [8] proved that the solution of the heat equation

$$\frac{\partial}{\partial t} f(x,t) = \sigma \Delta f(x,t) - V(x)f(x,t) , \tag{1.14}$$

which is the analogue of Schrödinger's equation when t is replaced by $-it$, σ being diffusion's constant, can be expressed

$$f(x,t) = \int e^{-\int_0^t V(\gamma(\tau)+x)d\tau} \varphi(\gamma(0)+x)dW(\gamma) \,, \tag{1.15}$$

where $dW(\gamma)$ is Wiener's measure for the Wiener, i.e. Brownian motion, process with variance σ^2, defined on continuous paths [3] $\gamma(\tau)$, $0 \leq \tau \leq t$, with $\gamma(t) = 0$. Hence (1.15) is an expectation with respect to the normal unit distribution indexed by the real Hilbert space of absolutely continuous functions $\gamma(\tau)$, with norm $\|\gamma\|^2 = \int(\frac{d\gamma}{d\tau})^2 d\tau$. From this we see that (1.15) can be formally rewritten as (1.13), with $\frac{i}{\hbar}S_t(\gamma)$ replaced by $-\frac{1}{2}\int_0^t \frac{1}{2\sigma}(\frac{d\gamma}{d\tau})^2 d\tau - \int_0^t V(\gamma(\tau))d\tau$. Thus (1.15) is actually a rigorous version of the correspondent of the Feynman path integral for the heat equation. This fact has been used [10] to provide a "definition by analytic continuation" of the Feynman path integral, in the sense that Feynman's path integral is then understood as the analytic continuation to purely imaginary t of the Wiener integral (1.15). The analogous continuation of the Wiener integral solution of the equation (1.14), with V replaced by iV, which corresponds to Schrödinger's equation with purely imaginary mass m, has been studied by Nelson [10] and allows to cover the case of some singular potentials. These definitions by analytic continuation, as well as the definition by the "sequential limit" (1.10), [4] have the disadvantage of being indirect in as much as they do not exhibit Feynman's solution (1.13) as an integral of the exponential of the action over a space of paths in physical space-time. On the other hand, as first pointed out by Cameron [10] and Daletskii [10], in relation to a remark in [5], the simple replacement in Wiener's measure of the variance by a complex one yields a complex measure with infinite total variation, which certainly makes impossible a definition of Feynman path integral as an integral with respect to this measure, such that every bounded continuous functional should be integrable. For further remarks on this measure see [12]. A definition of Feynman path integrals for non relativistic quantum mechanics, not involving analytic continuation as the one mentioned above, has been given by Ito [13]. We shall describe this definition in Section 2. Ito treated potentials $V(x)$ which are either Fourier transforms of bounded complex measures or of the form $c_\alpha x^\alpha$, with $\alpha = 1,2$, $c_2 > 0$. Ito's definition has been further discussed by Tarski [14]. Recently Morette - De Witt [15] has made a proposal for a definition of Feynman path integral which has some relations with Ito's definition but is more distributional rather than Hilbert space theoretical in character. The proposal suggests writing the Fourier transform of (1.13) as the "pseudomeasure"

$e^{-\frac{i}{2}W}$, looked upon as a distribution acting on the Fourier transform of $e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau}$, provided this exists, where W is the Fourier transform of Wiener's measure with purely imaginary variance. This proposal left open the classes of functions V for which it actually works. Such classes follow however from section 2 of the present work. Despite its, so far, incompleteness as to the class of allowed potentials, let us also mention a general attempt by Garczynski [16] to define Feynman path integrals as averages with respect to certain quantum mechanical Brownian motion processes, which generalize the classical ones. This approach has, incidentally, connections with stochastic mechanics [17], which itself would be worthwhile investigating in relation to the Feynman path formulation of quantum mechanics. 5)

Let us now make a correspondent brief historical sketch about the problem of the mathematical definition of Feynman path integrals in quantum field theory. They were introduced as heuristic tools by Feynman in [1] and applied by him to the derivation of the perturbation expansion in quantum electrodynamics. They have been used widely since then in the physical literature, see e.g. [18], also under the name of Feynman history integrals. We shall now shortly **give** their formal expression. For more details see, besides the original papers [1], also e.g. [18]. The classical formal action for the relativistic scalar boson field is $S(\varphi) = S_0(\varphi) + \int_{R^{n+1}} V(\varphi(\vec{x},t))d\vec{x}\,dt$, with

$$S_0(\varphi) = \frac{1}{2}\int_{R^{n+1}} \left[\left(\frac{d\varphi}{dt}\right)^2 - \sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 - m^2\varphi^2\right]d\vec{x}\,dt\,,$$

where φ is a function of $\vec{x}$, t, m is a non negative constant, the mass of the field, and V is the interaction. Similarly as in the case of a particle, the classical solutions of the equations of motion is given by Hamilton's principle of least action. The correspondent quantized system is formally characterized by the so called time ordered vacuum expectation values $G(\vec{x}_1,t_1,\ldots,\vec{x}_k,t_k)$, formally given, for $t_1 \leq \ldots \leq t_k$, $k = 1,2,\ldots$, by the expectations of the products $\Phi(\vec{x}_1,t_1)\ldots\Phi(\vec{x}_k,t_k)$ in the vacuum state, where $\Phi(\vec{x},t)$ is the quantum field (see e.g. [18],7) An heuristic expression for these quantities in terms of Feynman history integrals is

$$G(\vec{x}_1,t_1,\ldots,\vec{x}_k,t_k) = T\left(\int e^{iS(\varphi)}\,\varphi(\vec{x}_1,t_1)\ldots\varphi(\vec{x}_k,t_k)d\varphi\right),$$

where T is the so called time ordering operator and the integrals are thought as integrals over a suitable subset of real functions φ on R^{n+1}, see e.g. [18],7). A mathematical justification of this formula, or a related one, would actually provide a solution of the well known problem of the construction of relativistic quantum field theory. Somewhat in connection, in one way or the other, with this problem, a large body of theory on integration in function spaces has been developed since the fifties and we mention in particular the work by Friedrichs [19], Gelfand [20], Gross [21] and Segal [22] and their associates, see also e.g. [23]. With respect to the specific application to quantum field theory, more recently a study of models has been undertaken, see e.g. [24], in which either the relativistic interaction is replaced by an approximate one, with the ultimate goal of removing at a later stage the approximation, or physical space - time is replaced by a lower dimensional one. We find here methods which parallel in a sense those discussed above in relation with Schrödinger's equation and, in a similar way as in that case, we can put these methods in connection with the problem of giving meaning to Feynman path integral, although in this case the connection is even a more indirect one as it was in the non relativistic case. We mention however these methods for their intrinsic interest. The sequential approach, based on Lie-Kato-Trotter formula has been used especially in two space-time dimensional models particularly by Glimm, Jaffe and Segal [25]. The analytic continuation approach, in which time is replaced by imaginary time, is at the basis of the so called Euclidean-Markov quantum field theory, pursued vigorously by Symanzik [26] and Nelson [27] and applied particularly successfully, mostly in connection with the fundamental work of Glimm and Jaffe, for local relativistic models in two space-time dimensions, with polynomial [24] or exponential interactions [28],[6] and in three space-time dimensions with space cut-off [29],respectively in higher dimensions with ultraviolet cut-off interactions [30]. Much in the same way as for the heat, Schrödinger and stochastic mechanics equations,there are connections also with stochastic field theory [17], 4)-7).

Coming now to the Feynman history integrals themselves, it does not seem, to our knowledge, that any work has been done previous to our present work, as to their direct mathematical definition as integrals on a space of paths in physical space-time, except for the free case [14].

We shall now summarize briefly the content of the various sections of our work.

In section 2 we introduce the basic definition for oscillating integrals on a separable real Hilbert space, which we call Fresnel integrals, and we establish their properties. In section 3 this theory is applied to the definition of Feynman path integrals in non relativistic quantum mechanics. We prove that the heuristic Feynman path integral formula (1.13) for the solution of Schrödinger's equation can be interpreted rigorously as a Fresnel integral over a Hilbert space of continuous paths. In addition we derive corresponding formulae also for the wave operators and for the scattering operator. In section 4 we extend, in view of further applications, the definition of Fresnel integrals and give the properties of the new integral, called Fresnel integral relative to a given quadratic form. This theory is applied in section 5 to the definition of Feynman path integrals for the n-dimensional anharmonic oscillator and in sections 6 and 7 to the expression of expectations of functions of dynamical quantities of this anharmonic oscillator with respect to the ground state, respectively the Gibbs states [33] and quasifree states [34] of the correspondent harmonic oscillator. In section 8 we express the time invariant quasifree states on the Weyl algebra of an infinite dimensional harmonic oscillator by Feynman path integrals defined as Fresnel integrals in the sense of section 4, and this also provides a characterization of such states.

Finally, in section 9 we apply the results of section 8 to the study of relativistic quantum field theory. For the ultraviolet cut-off models mentioned above [ ] we express certain expectation values, connected with the time ordered vacuum expectation values, in terms of Feynman history integrals, again defined as Fresnel integrals relative to a quadratic form. We also derive the correspondent expressions for the expectations with respect to any invariant quasi-free state, in particular for the Gibbs states of statistical mechanics for quantum fields ([33]3)).

2. The Fresnel integral of functions on a separable real Hilbert space.

We consider first the case of the finite dimensional real Hilbert space R^n, with some positive definite scalar product (x,y). We shall use $|x|$ for the Hilbert norm of x, such that $|x|^2 = (x,x)$. Since $e^{\frac{i}{2}|x|^2}$ is a bounded continuous function it has a Fourier transform in the sense of tempered distributions and in fact

$$\int e^{\frac{i}{2}|x|^2} e^{i(x,y)} dx = (2\pi i)^{\frac{n}{2}} e^{-\frac{i}{2}|y|^2} , \qquad (2.1)$$

with $dx = dx_1, \ldots, dx_n$, where $x_i = (e_i, x)$, $e_1, \ldots, e_n$ being some orthonormal base in R^n with respect to the inner product $(,)$. For any function f in the Schwartz space $\mathcal{S}(R^n)$ we shall introduce for convenience the notation

$$\tilde{\int} f(x)dx = (2\pi i)^{-\frac{n}{2}} \int f(x)dx , \qquad (2.2)$$

so that $\tilde{\int} f(x)dx$ is proportional to the usual integral with a normalization factor that depends on the dimension. We get from (2.1) that, for any $\varphi \in \mathcal{S}(R^n)$,

$$\tilde{\int} e^{\frac{i}{2}|x|^2} \hat{\varphi}(x)dx = \int e^{-\frac{i}{2}|x|^2} \varphi(x)dx , \qquad (2.3)$$

where $\hat{\varphi}(x) = \int e^{i(x,y)} \varphi(y)dy$.

Let now $f(x)$ be the Fourier transform of a bounded complex measure μ, $\|\mu\| < \infty$, $f(x) = \int e^{i(x,y)} d\mu(y)$. We shall denote by $\mathcal{F}(R^n)$ the linear space of functions which are Fourier transforms of bounded complex measures. Since the space of bounded complex measures $\mathcal{M}(R^n)$ is a Banach algebra under convolution in the total variation norm $\|\mu\|$, we get that $\mathcal{F}(R^n)$ is a Banach algebra under multiplication in the norm $\|f\|_0 = \|\mu\|$ for $f(x) = \int e^{i(x,y)} d\mu(y)$. The elements in $\mathcal{F}(R^n)$ are bounded continuous functions and we have obviously $\|f\|_\infty \leq \|f\|_0$. For any $f \in \mathcal{F}(R^n)$ of the form

$$f(x) = \int e^{i(x,y)} d\mu(y)$$

we define

$$\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx = \int e^{-\frac{i}{2}|x|^2} d\mu(x) . \tag{2.4}$$

The right hand side is well defined since $e^{-\frac{i}{2}|x|^2}$ is bounded continuously and $\mu(x)$ is a bounded complex measure. (2.4) defines $\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx$ for all $f \in \mathcal{F}(R^n)$, and it follows from (2.3) that, for $f \in \mathcal{S}(R^n)$, $(2\pi i)^{n/2} \tilde{\int} e^{\frac{i}{2}|x|} f(x)dx$ is just the usual translation invariant integral $\int e^{\frac{i}{2}|x|^2} f(x)dx$ in R^n. Hence $(2\pi i)^{n/2} \tilde{\int}$ is an extension of the usual translation invariant integral on smooth functions. This extention is in a different direction than Lebesgue's extension, namely to the linear space of functions of the form $e^{\frac{i}{2}|x|^2} f(x)$ with $f \in \mathcal{F}(R^n)$. It follows from (2.4) that this extension is continuous in the sense that

$$|\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx| \leq \|f\|_0 . \tag{2.5}$$

Since $\tilde{\int}$ is an continuous extension of the normalized integral from S
$\mathcal{S}(R^n)$ to $\mathcal{F}(R^n)$, we shall say that $\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx$ is the normalized integral of the function $e^{\frac{i}{2}|x|^2} f(x)$. By (2.5) we have that

$$\mathcal{F}(f) = \tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx \tag{2.6}$$

is a bounded continuous functional on $\mathcal{F}(R^n)$. We shall call $\mathcal{F}(f)$ the Fresnel integral of f and we shall say that f is a Fresnel integrable function if $f \in \mathcal{F}(R^n)$.

Remark 1: We have chosen the denomination Fresnel integral for the continuous linear functional (2.6) because of the so called Fresnel integrals in the optical theory of wave diffraction, which are integrals of the form

$$\int_0^w e^{\frac{i\pi}{2}y^2} dy \ .$$

We now summarize the properties of the Fresnel integral which we have proved above.

Proposition 2.1.

The space $\mathcal{F}(R^n)$ of Fresnel integrable functions is a Banach-function-algebra in the norm $\|f\|_0$. The Fresnel integral

$$\mathcal{F}(f) = \tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx$$

is a continuous bounded linear functional on $\mathcal{F}(R^n)$ such that $|\mathcal{F}(f)| \leq \|f\|_0$ and normalized such that $\mathcal{F}(1) = 1$. For any $f(x) \in \mathcal{S}(R^n)$

$$\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx = (2\pi i)^{-\frac{n}{2}} \int e^{\frac{i}{2}|x|^2} f(x)dx \ .$$

It follows from the fact that $\mathcal{F}(R^n)$ is a Banach algebra that $\mathcal{F}(f_1,\ldots,f_k)$ is a continuous k-linear form on $\mathcal{F}(R^n)\times\ldots\times\mathcal{F}(R^n)$ such that

$$|\mathcal{F}(f_1,\ldots,f_k)| \leq \|f_1\|_0 \cdots \|f_k\|_0 .$$

Moreover sums and products of Fresnel integrable functions are again Fresnel integrable, and the composition with entire functions is also Fresnel integrable. □

We shall now define the normalized integral and the Fresnel integral in a real separable Hilbert space. So let $\mathcal{H}$ be a real separable Hilbert space with inner product (x,y) and norm $|x|$. $\mathcal{H}$ is then a separable metric group under addition. Let $\mathcal{M}(\mathcal{H})$ be the Banach space of bounded complex Borel-measures on $\mathcal{H}$. Let μ and ν be two elements in $\mathcal{M}(\mathcal{H})$. The convolution $\mu*\nu$ is defined by

$$\mu * \nu(A) = \int \mu(A-x)d\nu(x) \tag{2.7}$$

where A is a Borel set. It follows from the fact that $\mathcal{H}$ is a separable metric group that (2.7) is well defined. Moreover $\mathcal{M}(\mathcal{H})$ is in fact a topological semigroup under convolution in the weak topology. For a proof of this fact see reference [41], theorem 1.1 page 57. So that $\mu*\nu$ is simultaneously weakly continuous in μ and ν. From (2.7) we have, for any bounded continuous function f on $\mathcal{H}$, that

$$\int f(x)\, d(\mu*\nu)(x) = \int f(x+y)d\mu(x)d\nu(y) , \tag{2.8}$$

which gives that $\mu*\nu = \nu*\mu$ and $\|\mu*\nu\| \leq \|\mu\|\|\nu\|$, so that $\mathcal{M}(\ )$ is a commutative Banach algebra under convolution. We define $\mathcal{F}(\mathcal{H})$ as the space of bounded continuous functions on $\mathcal{H}$ of the form

$$f(x) = \int e^{\,i(x,y)}\, d\mu(y) ,$$

for some $\mu \in \mathcal{M}(\mathcal{H})$. The mapping $\mu \to f$ given by (2.8) is linear and also one-to-one. This is so because, if $f=0$, then the

μ-measure of any set of the form $\{y;\ (x,y) \geq \alpha\}$ is zero, from which it follows that the μ-measure of any closed convex set is zero. Therefore the μ-measure of any ball is zero, from which it follows that the μ-measure of any strongly measurable set is zero, which implies $\mu = 0$. Therefore introducing the norm $\|f\|_o = \|\mu\|$, we get that $\mu \to f$ is an isometry onto. It follows from (2.8) that convolution goes into product, so that $\mathcal{F}(\mathcal{H})$, with the norm $\|f\|_o$, is a Banach-function-algebra of continuous bounded functions. From (2.8) we also get that $\|f\|_\infty \leq \|f\|_o$. We now define the normalized integral on $\mathcal{H}$ by

$$\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx = \int e^{-\frac{i}{2}|x|^2} d\mu(x) , \tag{2.9}$$

for f given by (2.8). We also introduce the Fresnel integral of $f \in \mathcal{F}(\mathcal{H})$ by

$$\mathcal{F}(f) = \tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx . \tag{2.10}$$

We shall call $\mathcal{F}(\mathcal{H})$ the space of Fresnel integrable functions on $\mathcal{H}$.

<u>Proposition 2.2</u>.

The space $\mathcal{F}(\mathcal{H})$ of Fresnel integrable functions is a Banach-function-algebra in the norm $\|f\|_o$. The Fresnel integral $\mathcal{F}(f)$ is a continuous bounded linear functional on $\mathcal{F}(\mathcal{H})$ such that $|\mathcal{F}(f)| \leq \|f\|_o$ and normalized so that $\mathcal{F}(1) = 1$. It follows from the fact that $\mathcal{F}(\mathcal{H})$ is a Banach-algebra that sum and products of Fresnel integrable functions are again Fresnel integrable functions, and so are also compositions with entire functions.

If $f \in \mathcal{F}(\mathcal{H})$ is a finitely based function, i.e. there exists a finite dimensional orthogonal projection P in $\mathcal{H}$ such that $f(x) = f(Px)$ for all $x \in \mathcal{H}$, then

$$\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx = \tilde{\int}_{P\mathcal{H}} e^{\frac{i}{2}|x|^2} f(x)dx \ ,$$

where the normalized integral on the right hand side is the normalized integral on the finite dimensional Hilbert space $P\mathcal{H}$ defined previously. This could also be written

$$\mathcal{F}_{\mathcal{H}}(f) = \mathcal{F}_{P\mathcal{H}}(f) \ .$$

Proof: The first part is proved as in proposition 2.1. To prove the second part we use the definition

$$\tilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|x|^2} f(x)dx = \int_{\mathcal{H}} e^{-\frac{i}{2}|x|^2} d\mu(x) \ .$$

Now for $f(x)$ to be finitely based with base $P\mathcal{H}$ implies easely that μ has support contained in $P\mathcal{H}$, so that

$$\begin{aligned}\int_{\mathcal{H}} e^{-\frac{i}{2}|x|^2} d\mu(x) &= \int_{P\mathcal{H}} e^{-\frac{i}{2}|x|^2} d\mu(x) \\ &= \tilde{\int}_{P\mathcal{H}} e^{\frac{i}{2}|x|^2} f(x)dx \ .\end{aligned}$$

This then proves proposition 2. $\square$

It follows from its definition that $\mathcal{F}(\mathcal{H})$ is invariant under translations and orthogonal transformations of $\mathcal{H}$, and in fact the induced transformations in $\mathcal{F}(\mathcal{H})$ are isometries of $\mathcal{F}(\mathcal{H})$. It follows from the definition (2.9) of the normalized integral that it is invariant under orthogonal transformations of $\mathcal{H}$. Consider now a translation $x \to x+a$ of $\mathcal{H}$. The in-

duced transformation in $\mathcal{F}(\mathcal{H})$ is $f \to f_a$, where $f_a(x) = f(x+a)$.
If

$$f(x) = \int e^{i(x,y)} \, d\mu(y)$$

then

$$f_a(x) = \int e^{i(x,y)} e^{i(a,y)} \, d\mu(y) .$$

By the definition (2.9) we then have

$$\tilde{\int} e^{\frac{i}{2}|x|^2} f_a(x)dx = \int e^{-\frac{i}{2}|x|^2} e^{i(a,x)} \, d\mu(x)$$

$$= e^{\frac{i}{2}|a|^2} \int e^{-\frac{i}{2}|x-a|^2} \, d\mu(x)$$

$$= e^{\frac{i}{2}|a|^2} \int e^{-\frac{i}{2}|x|^2} \, d\mu(x+a) .$$

Now

$$\int e^{i(x,y)} d\mu(y+a) = e^{-i(x,a)} \int e^{i(x,y)} d\mu(y) = e^{-i(x,a)} f(x) .$$

From the relation

$$e^{-i(x,a)} f(x) = \int e^{i(x,y)} \, d\mu(y+a)$$

we get by (2.9)

$$\tilde{\int} e^{\frac{i}{2}|x|^2} e^{-i(x,a)} f(x)dx = \int e^{-\frac{i}{2}|x|^2} \, d\mu(x+a) .$$

Hence

$$\tilde{\int} e^{\frac{i}{2}|x-a|^2} f(x)dx = e^{\frac{i}{2}|a|^2} \int e^{-\frac{i}{2}|x|^2} \, d\mu(x+a)$$

$$= \tilde{\int} e^{\frac{i}{2}|x|^2} f_a(x)dx ,$$

which proves that the normalized integral is also invariant under translations of $\mathcal{H}$. We state these results in the following proposition.

Proposition 2.3.

Let the group of Euclidean transformations $E(\mathcal{H})$ be the group of transformations $x \to Ox + a$, where $a \in \mathcal{H}$. and O is an orthogonal transformation of $\mathcal{H}$ onto $\mathcal{H}$. Then the space of Fresnel integrable functions $\mathcal{F}(\mathcal{H})$ is invariant under $E(\mathcal{H})$, and $e(\mathcal{H})$ is in fact a group of isometries of $\mathcal{F}(\mathcal{H})$. Moreover the normalized integral is invariant under the transformations in $E(\mathcal{H})$. $\square$

We shall now prove the analogue of the the Fubini theorem for the normalized integral. So let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then $f \in \mathcal{F}(\mathcal{H})$ defines a continuous function $f(x_1,x_2)$ on $\mathcal{H}_1 \times \mathcal{H}_2$ by $f(x_1 \oplus x_2) = f(x_1,x_2)$. For fixed $x_2 \in \mathcal{H}_2$, $f(x_1,x_2) \in \mathcal{F}(\mathcal{H}_1)$. To see this, we note that we have by definition

$$f(x) = \int e^{i(x,y)} d\mu(y) .$$

Since $\|x_1 \oplus x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ we get that $\mathcal{H}$ and $\mathcal{H}_1 \times \mathcal{H}_2$ are equivalent metric spaces, so that $d\mu(y)$ is actually a measure on the product space $\mathcal{H}_1 \times \mathcal{H}_2$ with the product measure structure. We shall write this measure on $\mathcal{H}_1 \times \mathcal{H}_2$ as $d\mu(y_1,y_2)$.

Hence

$$f(x_1,x_2) = \int e^{i(x_1,y_1)} e^{i(x_2,y_2)} d\mu(y_1,y_2) . \tag{2.11}$$

Consider now the measure $\mu_{x_2} \in \mathcal{M}(\mathcal{H}_1)$ defined by

$$\int_{\mathcal{H}_1} \varphi(y_1) d\mu_{x_2}(y_1) = \int \varphi(y_1)\, e^{i(x_2,y_2)} d\mu(y_1,y_2) . \tag{2.12}$$

By the usual Fubini theorem we then have that

$$f(x_1,x_2) = \int_{\mathcal{H}_1} e^{i(x_1,y_1)} d\mu_{x_2}(y_1) . \tag{2.13}$$

This proves that, for fixed $x_2 \in \mathcal{H}_2$, $f(x_1,x_2) \in \mathcal{F}(\mathcal{H}_1)$. Hence the normalized integral

$$g(x_2) = \widetilde{\int_{\mathcal{H}_1}} e^{\frac{i}{2}|x_1|^2} f(x_1,x_2)dx_1 \qquad (2.14)$$

is well defined. We shall now see that $g(x_2) \in \mathcal{F}(\mathcal{H}_2)$. By the definition of the normalized integral and (2.13) we have that

$$\widetilde{\int_{\mathcal{H}_1}} e^{\frac{i}{2}|x_1|^2} f(x_1,x_2)dx_1 = \int_{\mathcal{H}_1} e^{-\frac{i}{2}|y_1|^2} d\mu_{x_2}(y_1) \qquad (2.15)$$

and by (2.12) this is equal to

$$\int e^{-\frac{i}{2}|y_1|^2} e^{i(x_2,y_2)} d\mu(y_1,y_2) . \qquad (2.16$$

Hence

$$g(x_2) = \int_{\mathcal{H}_2} e^{i(x_2,y_2)} d\nu(y_2) , \qquad (2.17)$$

where $\nu \in \mathcal{M}(\mathcal{H}_2)$ is defined by

$$\int_{\mathcal{H}_2} \varphi(y_2)d\nu(y_2) = \int e^{-\frac{i}{2}|y_1|^2} \varphi(y_2) d\mu(y_1,y_2) . \qquad (2.18)$$

This then proves that $g(x_2) \in \mathcal{F}(\mathcal{H}_2)$, and the normalized integral

$$\widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2}|x_2|^2} g(x_2)dx_2 \qquad (2.19)$$

is well defined.

By (2.17) and the definition of the normalized integral we have that

$$\tilde{\int\limits_{\mathcal{H}_2}} e^{\frac{i}{2}|x_2|^2} g(x_2)dx_2 = \int\limits_2 e^{-\frac{i}{2}|y_2|^2} d\nu(y_2)$$

which by Fubini and (2.18) is equal to

$$\int e^{-\frac{i}{2}|y_1|^2} e^{-\frac{i}{2}|y_2|^2} d\mu(y_1,y_2)$$

$$= \tilde{\int\limits_{\mathcal{H}}} e^{\frac{i}{2}|x|^2} f(x)dx \ .$$

We have now proven the following proposition, which we may also call the Fubini theorem for the normalized integral.

<u>Proposition 2.4</u>. (Fubini theorem)

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be the orthogonal sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For $f(x) \in \mathcal{F}(\mathcal{H})$ set $f(x_1,x_2) = f(x_1 \oplus x_2)$ with $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$. Then for fixed x_2, $f(x_1,x_2)$ is in $\mathcal{F}(\mathcal{H}_1)$ and

$$g(x_2) = \tilde{\int} e^{\frac{i}{2}|x_1|^2} f(x_1,x_2)dx_1$$

is in $\mathcal{F}(\mathcal{H}_2)$. Moreover

$$\tilde{\int\limits_{\mathcal{H}_2}} e^{\frac{i}{2}|x_2|^2} g(x_2)dx_2 = \tilde{\int\limits_{\mathcal{H}_2}} e^{\frac{i}{2}|x_2|^2} \Big(\tilde{\int\limits_{\mathcal{H}_1}} e^{\frac{i}{2}|x_1|^2} f(x_1,x_2)dx_1\Big)dx_2$$

$$= \tilde{\int\limits_{\mathcal{H}}} e^{\frac{i}{2}|x|^2} f(x)dx \ . \qquad \square$$

Remark: The normalized integral on a separable Hilbert space is the same as the functional F defined by Ito in [13].

$$\tilde{\int} e^{\frac{i}{2}|x|^2} f(x)dx = F(e^{\frac{i}{2}|x|^2} f) \ .$$

Ito takes a completely different definition, he defines namely F for $f \in \mathcal{F}(\mathcal{H})$ by

$$F(e^{\frac{i}{2}|x|^2} f) = \lim_V \prod_{j=1}^{\infty} (1-i\lambda_j)^{\frac{1}{2}} E(e^{\frac{i}{2}|x|^2} f;a,V) \ ,$$

where $E(g;a,V)$ is the expectation of g with respect to the Gaussian measure with mean $a \in \mathcal{H}$ and covariance operator V, where V is a strictly positive definite symmetric trace class operator on $\mathcal{H}$ with eigenvalues λ_j, such that $\sum_{j=1}^{\infty} \lambda_j < \infty$. The limit is taken along the directed system of all strictly positive definite trace class operators with the direction given by the relation $<$, where $V_1 < V_2$ if and only if $V_2 - V_1$ is positive. Ito proves that this limit exists and is independent of a and moreover that it is invariant under nearly isometric transformations, in the sense that

$$F(e^{\frac{i}{2}|Cx|^2} f(Cx)) = J(C)^{-1} F(e^{\frac{i}{2}|x|^2} f) \ ,$$

where $Cx = Ax + b$, with $b \in \mathcal{H}$ and A a one-to-one map of $\mathcal{H}$ such that

$$\operatorname{tr}([(A^*A)^{\frac{1}{2}} - 1]^{\alpha/2}) < \infty$$

for some $\alpha < 1$. $J(C)$ is defined by $J(C) = \prod_{j=1}^{\infty} (1+\alpha_j)$, where α_j are the eigenvalues of $(A^*A)^{\frac{1}{2}} - 1$.

Instead of Ito's definition we have used Parseval relation (2.9) as a definition for the normalized integral because we shall later need a generalization of the normalized integral to spaces with indefinite metric. These generalizations will be very natural in our setting and in fact defined again by a sort of Parseval relation. We feel also that the definition of the normalized integral by the Parseval relation (2.9) gives a nice and simple introduction to the properties of the normalized integral.

We also want to point out that the first part of the next section, i.e. the formula for the finite time transition amplitude, was first derived by Ito [13], but we shall give the proof of it here partly for the sake of completeness and also because our proof is independent of Ito's, and it will later on be extended to cover different situations.

3. The Feynman path integral in potential scattering.

In this section we consider the Schrödinger equation for a quantum mechanical particle in R^n under the influence of a potential $V(x)$. The Schrödinger equation for the wavefunction $\psi(x,t)$ is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta\psi + V\psi \tag{3.1}$$

where m is the mass of the particle, $\hbar$ is Planck's constant and Δ is the Laplacian $\Delta\psi = \sum_{i=1}^{n} \frac{\partial^2\psi}{\partial x_i^2}$. For typographical reasons we choose units for time and length such that $\hbar = 1$, in which case we get

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \Delta\psi + V\psi . \tag{3.2}$$

$H_o = -\frac{1}{2m}\Delta$ is a self adjoint operator in $L_2(R^n)$ on its natural domain of definition. In what follows V will be a bounded continuous real function on R^n, hence $H = H_o + V$ is also a self adjoint operator with the same domain as H_o. The solution of the initial value problem for (3.2) is therefore given by

$$\psi(x,t) = (e^{-itH}\varphi)(x) \tag{3.3}$$

with initial data $\varphi \in L_2(R^n)$, where e^{-itH} is the unitary group in $L_2(R^n)$ generated by $-H$. We shall now express (3.3) as a socalled Feynman path integral. In fact (3.3) will be given by the normalized integral of e^{iS}, where S is the classical action, over all path's for the particle ending at x at time t. This expression was suggested by Feynman and proved by Ito [13], and we shall therefore call it the Feynman-Ito formula. It is the correspondent for the Schrödinger equation of the Feynman-Kac formula for the heat equation.

It is well known that e^{-itH} can be expanded in powers of V. We have namely that

$$\frac{d}{dt} e^{itH_o} e^{-itH} = -i\, e^{itH_o} V e^{-itH}$$
$$= -i\, V(t)\, e^{itH_o} e^{-itH} ,$$

where $V(t) = e^{itH_o} V e^{-itH_o}$. Integrating this formula we get

$$e^{itH_o} e^{-itH} = 1 - i \int_0^t V(t_1)\, e^{it_1 H_o} e^{-it_1 H} dt_1 . \tag{3.4}$$

By iteration we have then

$$e^{itH_o} e^{-itH} = \sum_{n=0}^{\infty} (-i)^n \int_{t \geq t_1 \geq \ldots \geq t_n \geq 0} \cdots \int V(t_1) \ldots V(t_n) dt_1 \ldots dt_n , \tag{3.5}$$

which by the norm boundedness of V is obviously norm convergent for all t. By substitution under the integrals we get from (3.5)

$$e^{-itH} = \sum_{n=0}^{\infty} (-i)^n \int_{0 \leq t_1 \ldots \leq t_n \leq t} \cdots \int e^{-itH_o} V(t_n) \ldots V(t_1) dt_1 \ldots dt_n , \tag{3.6}$$

or more explicitely

$$e^{-itH} = \sum_{n=0}^{\infty} (-i)^n \int_{0 \leq t_1 \leq \ldots \leq t_n \leq t} \cdots \int e^{-i(t-t_n)H_o} V e^{-i(t_n - t_{n-1})H_o} \ldots$$
$$\ldots e^{-i(t_2 - t_1)H_o} V e^{-it_1 H_o} dt_1 \ldots dt_n . \tag{3.7}$$

We shall now assume that V is of the form

$$V(x) = \int_{R^n} e^{i\alpha x} d\mu(x) \tag{3.8}$$

and that

$$\varphi(x) = \int_{R^n} e^{i\alpha x} d\nu(\alpha) , \tag{3.9}$$

where $\alpha x = \sum_{i=1}^{n} \alpha_i x_i$, for some bounded complex measures μ and ν on R^n. Since $e^{i\alpha x}$ is a generalized eigenfunction for H_o, with

$$e^{-itH_o} e^{i\alpha x} = e^{-\frac{it}{2m}\alpha^2} e^{i\alpha x} \tag{3.10}$$

we get by substitution of (3.8), (3.9) and (3.7) in (3.3) that

$$\psi(x,t) = \sum_{n=0}^{\infty} (-i)^n \int_{0\le t_1 \le \dots \le t_n \le t} \!\!\cdots\!\! \int \cdot$$

$$e^{-\frac{i}{2m}[(t-t_n)(\alpha_o+\dots+\alpha_n)^2+(t_n-t_{n-1})(\alpha_o+\dots+\alpha_{n-1})^2+\dots+(t_2-t_1)(\alpha_o+\alpha_1)^2+t_1\alpha_o^2]} \cdot$$

$$\cdot\, e^{i(\sum_{j=0}^{n}\alpha_j)x} \, d\nu(\alpha_o) \prod_{j=1}^{n} (d\mu(\alpha_j)dt_j) \, . \tag{3.11}$$

Introducing the notation $t_o = 0$ we may simplify the exponent in (3.11) and we get

$$\psi(x,t) = \sum_{n=0}^{\infty} (-i)^n \int_{0\le t_1 \le \dots \le t_n \le t} \!\!\cdots\!\! \int e^{-\frac{i}{2m} \sum_{j,k=0}^{n} (t-t_j \vee t_k)\alpha_j \alpha_k} e^{i(\sum_{j=0}^{n}\alpha_j)x} \cdot$$

$$\cdot\, d\nu(\alpha_o) \prod_{j=1}^{n} (d\mu(\alpha_j)dt_j) \tag{3.12}$$

where $\sigma \vee \tau = \max\{\sigma,\tau\}$.

By the symmetry of the integrand we have then

$$\psi(x,t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \!\!\cdots\!\! \int_0^t \!\! \int \!\!\cdots\!\! \int e^{-\frac{i}{2m} \sum_{j,k=0}^{n} (t-t_j \vee t_k)\alpha_j \alpha_k} e^{i(\sum_{j=0}^{n}\alpha_j)x} \cdot$$

$$d\nu(\alpha_o) \prod_{j=1}^{n} (d\mu(\alpha_j)dt_j) \, . \tag{3.13}$$

Now $G_{ij}(\sigma,\tau) = (t-\sigma\vee\tau)\delta_{ij}$ is the Green's function or the kernel of the inverse operator of $-\frac{d^2}{d\tau^2}$ as a self adjoint operator on $L_2([0,t];R^n)$ with boundary conditions $\frac{du}{d\tau}(0) = u(t) = 0$.

Hence if we introduce the real Hilbert space $\mathcal{H}$ of real continuous functions $\gamma(\tau)$ from $[0,t]$ to R^n such that $\frac{d\gamma}{d\tau} \in L_2([0,t];R^n)$ and $\gamma(t) = 0$ with inner product $(\gamma_1,\gamma_2) = m \int_0^t \frac{d\gamma_1}{d\tau} \cdot \frac{d\gamma_2}{d\tau} = d\tau$, then $\gamma_{(\sigma,i)}(\tau,j) = G_{ij}(\sigma,\tau) = (t-\sigma\vee\tau)\delta_{ij}$ is in $\mathcal{H}$ for any (σ,i), and for any $\gamma \in \mathcal{H}$ we have

$$\gamma(\sigma,i) = m(\gamma,\gamma_{(\sigma,i)}) \ . \qquad (3.14)$$

It follows from (3.14) that the functions on $\mathcal{H}$ given by

$$\varphi(\gamma(0)+x) = \int_{R^n} e^{i\gamma(0)\cdot\alpha}\, e^{i\alpha x}\, d\nu(\alpha)$$

and

$$\int_0^t V(\gamma(\tau)+x)d\tau = \int_0^t \int_{R^n} e^{i\gamma(\tau)\cdot\alpha}\cdot e^{ix\alpha}\, d\mu(\alpha)d\tau$$

both are Fourier transforms of bounded measures on $\mathcal{H}$ so that both functions are in $\mathcal{F}(\mathcal{H})$, where $\mathcal{F}(\mathcal{H})$ is the space of Fresnel integrable functions on $\mathcal{H}$ defined in the previous section. By proposition 2 of the previous section we therefore have that the continuous function $f(\gamma)$ on $\mathcal{H}$ given by

$$f(\gamma) = e^{-i\int_0^t V(\gamma(\tau)+x)d\tau}\, \varphi(\gamma(0)+x) \qquad (3.15)$$

is in $\mathcal{F}(\mathcal{H})$. Hence the normalized integral $\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma$ is well defined. Since the exponent is invariant under the transformation $\gamma \to \gamma - x$, it is natural to introduce also the following two notations for this integral:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma \equiv \widetilde{\int_{\gamma(t)=0}} e^{\frac{i}{2}m\int_0^t |\frac{d\gamma}{d\tau}|^2 d\tau} f(\gamma)d\gamma \qquad (3.16)$$

$$\equiv \widetilde{\int_{\gamma(t)=x}} e^{\frac{i}{2}m\int_0^t |\frac{d\gamma}{d\tau}|^2 d\tau} f(\gamma-x)d\gamma \ , \qquad (3.17)$$

where $(\gamma - x)(\tau) = \gamma(\tau) - x$.

We shall now compute the normalized integral (3.16), with $f(\gamma)$ given by (3.15). By proposition 2 of the previous section we have that

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|\gamma|^2} (\int_0^t V(\gamma(\tau)+x)d\tau)^n \varphi(\gamma(0)+x)d\gamma$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|\gamma|^2} [\int_0^t \cdots \int_0^t \int \cdots \int e^{i(\sum_{j=1}^n \gamma(t_j)\alpha_j + \gamma(0)\alpha_o)} \tag{3.18}$$

$$e^{ix\alpha_o} d\nu(\alpha_o) \prod_{j=1}^n e^{ix\alpha_j} d\mu(\alpha_j)dt_j]d\gamma .$$

Now, by expressing $\sum_{j=1}^n \gamma(t_j)\alpha_j + \gamma(0)\alpha_o$ as a scalar product in $\mathcal{H}$ by the formula (3.14), we see that the n'th term in the sum in (3.18) is the normalized integral of the Fourier transform of a bounded measure on $\mathcal{H}$. Hence by the definition (2.9) of the normalized integral we get that (3.18) is equal to

$$\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \cdots \int_0^t \int \cdots \int e^{-\frac{i}{2m} \sum_{j,k=0}^n \alpha_j G(t_j,t_k)\alpha_k} e^{i(\sum_{j=0}^n \alpha_j)x} d\nu(\alpha_o) \prod_{j=1}^n d\mu(\alpha_j)dt_j, \tag{3.19}$$

where we have introduced the notation $t_o = 0$. By the definition of the matrix $G(\sigma,\tau)$ we see that (3.19) is equal to (3.13). Hence we have proved the Feynman-Ito formula, namely using the notation (3.17):

$$\psi(x,t) = \tilde{\int_{\gamma(t)=x}} e^{\frac{im}{2}\int_0^t |\frac{d\gamma}{d\tau}|^2 d\tau} \cdot e^{-i\int_0^t V(\gamma(\tau))d\tau} \varphi(\gamma(0))d\gamma . \tag{3.20}$$

Introducing the classical action along the path γ in the time interval $[0,t]$

$$S_t(\gamma) = \int_0^t \frac{m}{2}\left|\frac{d\gamma}{d\tau}\right|^2 d\tau - \int_0^t V(\gamma(\tau))d\tau \quad ,$$

the formula (3.20) may be written in more compact notations as

$$\psi(x,t) = \tilde{\int\limits_{\gamma(t)=x}} e^{iS_t(\gamma)} \varphi(\gamma(0))d\gamma \ . \tag{3.21}$$

With units such that $\hbar \neq 1$ we easely get the formula

$$\psi(x,t) = \tilde{\int\limits_{\gamma(t)=x}} e^{i/\hbar\, S_t(\gamma)} \varphi(\gamma(0))d\gamma \tag{3.22}$$

for the solution of the Schrödinger equation (3.1). We formulate this result, which was first established by Ito, in the following theorem

Theorem 3.1 (The Feynman-Ito formula)

Let V and φ be Fourier transforms of bounded complex measures in R^n. Let $\mathcal{H}$ be the real Hilbert space of continuous paths γ from $[0,t]$ to R^n such that $\gamma(t) = 0$ and $\frac{d\gamma}{d\tau} \in L_2([0,t];R^n)$ with inner product $(\gamma_1,\gamma_2) = \frac{m}{\hbar}\int_0^t (\frac{d\gamma_1}{d\tau}\cdot\frac{d\gamma_2}{d\tau})d\tau$.
Then

$$f(\gamma) = e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau)+x)d\tau} \varphi(\gamma(0)+x)$$

is in $\mathcal{F}(\mathcal{H})$, the space of Fresnel integrable functions on $\mathcal{H}$ and the solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$$

with boundary condition $\psi(x,0) = \varphi(x)$ is given by the normalized integral

$$\psi(x,t) = \tilde{\int\limits_{\mathcal{H}}} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma \ ,$$

i.e.

$$\psi(x,t) = \int\limits_{\gamma(t)=x}^{\sim} e^{\frac{im}{2\hbar}\int_0^t |\frac{d\gamma}{d\tau}|^2 d\tau} \cdot e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau} \varphi(\gamma(0))d\gamma \ . \quad \square$$

We shall now proceed to study the wave operator and the scattering operator in terms of Feynman path integrals. Let us again use units where $\hbar = 1$. Let us also, for typographically simplicity, assume that $m = 1$.

The wave operators $W_\pm$ are defined by

$$W_\pm = \underset{t \to \pm\infty}{\text{st. lim}}\ e^{-itH} e^{itH_o} , \tag{3.23}$$

whenever these limits exist.[1)] It is well known that these limits exist for a wide range of potentials $V(x)$ that fall off sufficiently fast. In what follows we shall assume that $V(x)$ is such that the limits (3.23) exist. So that

$$V(x) = \int_{R^n} e^{i\alpha x}\, d\mu(\alpha)$$

and for instance[2)]

$$|V(x)| \leq C(1+|x|)^{-1-\varepsilon}$$

(see for example [43]).

By expanding $e^{-isH_o} e^{-i(t-s)H} e^{itH_o}$ in powers of V in a similar manner as in (3.5) we get, with $V(t) = e^{-itH_o} V e^{itH_o}$, that

$$e^{-isH_o} e^{-i(t-s)H} e^{itH_o} = \sum_{n=0}^{\infty} (-i)^n \int\limits_{s \leq t_1 \leq \ldots \leq t_n \leq t} \!\!\ldots\!\! \int V(t_1)\ldots V(t_n)dt_1\ldots dt_n,$$

where the sum is norm convergent for all s and t,

i.e.

$$e^{-isH_0} e^{-i(t-s)H} e^{itH_0} = \sum_{n=0}^{\infty} (-i)^n \int\limits_{s\leq t_1\leq \dots <t_n\leq t}\cdots\int e^{-it_1H_0} V e^{-i(t_2-t_1)H_0} \cdots$$

$$\cdots e^{-i(t_n-t_{n-1})H_0} V e^{it_nH_0} dt_1 \dots dt_n .$$

With

$$\varphi(x) = \int e^{i\beta x} d\nu(\beta)$$

we get then

$$(e^{-isH_0} e^{-i(t-s)H} e^{itH_0} \varphi)(x) = \sum_{n=0}^{\infty} (-i)^n \int\limits_{s\leq t_1\leq \dots \leq t_n\leq t}\cdots\int$$

$$e^{-\frac{i}{2}[t_1(\alpha_1+\dots+\alpha_n+\beta)^2 + (t_2-t_1)(\alpha_2+\dots+\beta)^2+\dots+(t_n-t_1)(\alpha_n+\beta)^2 - t_n\beta^2]} \cdot \tag{3.24}$$

$$\cdot\, e^{i(\sum_{j=1}^{n}\alpha_j+\beta)x} d\nu(\beta) \prod_{i=1}^{n} (d\mu(\alpha_i)dt_i) .$$

We have

$$t_1(\alpha_1+\dots+\alpha_n+\beta)^2 + (t_2-t_1)(\alpha_2+\dots+\beta)^2+\dots+(t_n-t_{n-1})(\alpha_n+\beta)^2 - t_n\beta^2 \tag{3.25}$$

$$= t_1\alpha_1^2 + 2t_1\alpha_1(\alpha_2+\dots+\alpha_n+\beta) + t_2\alpha_2^2 + 2t_2\alpha_2(\alpha_3+\dots+\alpha_n+\beta)+\dots+t_n\alpha_n^2+2t_n\alpha_n\beta$$

$$= \sum_{ij=1}^{n} t_i \wedge t_j\alpha_i\alpha_j + 2\beta\sum_{i=1}^{n} t_i\alpha_i , \tag{3.26}$$

where $s \wedge t = \min\{s,t\}$.

If we introduce $\delta = -\sum_{i=1}^{n}\alpha_i - \beta$, (3.25) may also be written

$$t_1\delta^2 + (t_2-t_1)(\delta+\alpha_1)^2+\dots+(t_n-t_{n-1})(\delta+\alpha_1+\dots+\alpha_{n-1})^2 - t_n(\delta+\alpha_1+\dots+\alpha_n)^2$$

$$= - t_1\alpha_1^2 - 2t_1\delta\alpha_1 - t_2\alpha_2^2 - 2t_2\alpha_2(\delta+\alpha_1)+\dots -t_n\alpha_n^2 - 2t_n\alpha_n(\delta+\alpha_1+\dots+\alpha_{n-1})$$

$$= -\sum_{ij} t_i \vee t_j\alpha_i\alpha_j - 2\delta\sum_{i=1}^{n} t_i\alpha_i , \tag{3.27}$$

where $s \vee t = \max\{s,t\}$.

By the identity of (3.25) with (3.26) and (3.27) and the fact that $|s-t| = s \vee t - s \wedge t$ we have that (3.25) is also equal to

$$\sum_{ij=1}^{n} -\tfrac{1}{2}\,|t_i - t_j|\,\alpha_i\alpha_j + (\beta-\delta)\sum_{i=1}^{n} t_i\alpha_i \;. \tag{3.28}$$

If $s = 0$ and $t > 0$ we get from (3.24) and (3.25) that

$$(e^{-itH}\, e^{itH_0}\,\varphi)(x) = \sum_{n=0}^{\infty} (-i)^n \int_{0\le t_1 \le \ldots \le t_n \le t}\!\!\!\ldots\int \cdot$$

$$e^{-\frac{i}{2}\sum_{jk=1}^{n} t_j\wedge t_k \alpha_j\alpha_k - i\beta\sum_{j=1}^{n} t_j\alpha_j + i\left(\sum_{j=1}^{n}\alpha_j+\beta\right)x}\, d\nu(\beta)\prod_{j=1}^{n} d\mu(\alpha_j)\,dt_j \;. \tag{3.29}$$

By the substitution $t_i \to -t_i$, $i = 1,\ldots,n$ we get that (3.29 is equal to

$$\sum_{n=0}^{\infty} (-i)^n \int_{-t\le t_n \le \ldots \le t_1 \le 0}\!\!\!\ldots\int e^{-\frac{i}{2}\sum_{jk=1}^{n} -t_j\vee t_k \alpha_j\alpha_k + i\beta\sum_{j=1}^{n} t_j\alpha_j + i\left(\sum_{j=1}^{n}\alpha_j+\beta\right)x}$$

$$d\nu(\beta)\prod_{j=1}^{n} d\mu(\alpha_j)\,dt_j \;,$$

which, by the symmetry of the integrand, gives

$$(e^{-itH}\, e^{itH_0}\,\varphi)(x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-t}^{0}\!\!\ldots\int\int_{-t}^{0}\!\!\ldots\int \cdot$$

$$e^{-\frac{i}{2}\sum_{jk=1}^{n} -t_j\vee t_k \alpha_j\alpha_k + i\beta\sum_{j=1}^{n} t_j\alpha_j + i\left(\sum_{j=1}^{n}\alpha_j+\beta\right)x}\, d\nu(\beta)\prod_{j=1}^{n} d\mu(\alpha_j)\,dt_j \;. \tag{3.30}$$

Consider now the separable real Hilbert space $\mathcal{H}_-$ of continuous functions γ from $[-\infty,0]$ to R^n such that $\gamma(0) = 0$ and $\frac{d\gamma}{d\tau}$ is in $L_2([-\infty,0];R^n)$ with norm given by

$$|\gamma|^2 = \int_{-\infty}^{0} \left|\frac{d\gamma}{d\tau}\right|^2 d\tau \ . \tag{3.31}$$

We have that $\gamma_{(s,i)}(t,j) = -s \vee t \cdot \delta_{ij}$ is in $\mathcal{H}_-$ and for all $\gamma \in \mathcal{H}_-$ we have

$$(\gamma, \gamma_{(s,i)}) = \gamma(s,i) \ . \tag{3.32}$$

From this it follows that

$$f(\gamma) = \int_{-t}^{0} V(\gamma(\tau)+\beta\tau+x)d\tau = \int_{-t}^{0}\int e^{i\alpha\gamma(\tau)} \cdot e^{i\alpha(\beta\tau+x)} \, d\mu(\alpha)d\tau \tag{3.33}$$

is in $\mathcal{F}(\mathcal{H}_-)$. Hence, by proposition 2.2 of section 2, $e^{-if(\gamma)}$ again in $\mathcal{F}(\mathcal{H}_-)$ and the normalized integral

$$\tilde{\int_{\mathcal{H}_-}} e^{\frac{i}{2}|\gamma|^2} \, e^{-if(\gamma)} \, d\gamma \tag{3.34}$$

is well defined. By using (3.32) we may compute explicitely the normalized integral (3.34) in the same way as in the proof of theorem 3.1 and we get

$$\tilde{\int_{\mathcal{H}_-}} e^{\frac{i}{2}|\gamma|^2} \, e^{-if(\gamma)} \, d\gamma =$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-t}^{0}\cdots\int_{-t}^{0} \int\cdots\int e^{-\frac{i}{2}\sum_{jk=1}^{n} -t_j \vee t_k \alpha_j \alpha_k + i\beta\sum_{j=1}^{n} t_j\alpha_j + i(\sum_{j=1}^{n}\alpha_j)x} \cdot \prod_{j=1}^{n} d\mu(\alpha_j)dt_j \ . \tag{3.35}$$

Introducing now the notation

$$W_t(x,\beta) = \tilde{\int_{\mathcal{H}_-}} e^{\frac{i}{2}\int_{-\infty}^{0}\left|\frac{d\gamma}{d\tau}\right|^2 d\tau} \cdot e^{-i\int_{-t}^{0} V(\gamma(\tau)+\beta\tau+x)d\tau} \, d\gamma \tag{3.36}$$

we get from (3.30) that

$$(e^{-itH} e^{itH_0} \varphi)(x) = \int W_t(x,\beta)\, e^{i\beta x}\, d\nu(\beta) , \tag{3.37}$$

where

$$\varphi(x) = \int e^{ix\beta}\, d\nu(\beta) . \tag{3.38}$$

Since φ is in $L_2(R^n)$, which implies that $d\nu$ belongs to $L_1 \cap L_2$, we have, by the assumptions on the potential V , that (3.37) converges strongly. Hence, as $t \to \infty$, $W_t(x,\beta)$ converges in the strong topology of operators on $L_2(R^n)$ to a limit $W_+(x,\beta)$, which by (3.37) satisfies

$$(W_+\varphi)(x) = \int W_+(x,\beta)\, e^{i\beta x}\, d\nu(\beta) . \tag{3.39}$$

By the physical interpretation of the wave operators, $W_+(x,\beta)$ as a function of x is the wave function at time zero of the quantum mechanical particle with asymptotic momentum β as $t \to -\infty$. If $m \neq 1$ and $\hbar \neq 1$ one finds easely, by following the previous calculation, that

$$W_t(x,\beta) = \tilde{\int_{\mathcal{H}_-}} e^{\frac{im}{2\hbar}\int_{-\infty}^{0} |\frac{d\gamma}{d\tau}|^2 d\tau} \cdot e^{-\frac{i}{\hbar}\int_{-t}^{0} V(\gamma(\tau)+ \frac{\beta}{m}\tau+x)\, d\tau}\, d\gamma, \tag{3.40}$$

and we recall that a particle of momentum β has the classical velocity $\frac{\beta}{m}$. Hence we get the formula for the $W_+(x,\beta)$ of (3.39):

$$W_+(x,mv) = \lim_{t\to\infty} \tilde{\int_{\mathcal{H}_-}} e^{\frac{im}{2\hbar}\int_{-\infty}^{0} |\frac{d\gamma}{d\tau}|^2 d\tau} \cdot e^{-\frac{i}{\hbar}\int_{-t}^{0} V(\gamma(\tau)+v\tau+x)d\tau}\, d\gamma . \tag{3.41}$$

We shall also write this formula as an improper normalized integral

$$W_+(x,mv) = \widetilde{\int_{\mathcal{H}_-}} e^{\frac{im}{2\hbar}\int_{-\infty}^{0}\left|\frac{d\gamma}{d\tau}\right|^2 d\tau} \cdot e^{-\frac{i}{\hbar}\int_{-\infty}^{0} V(\gamma(\tau)+v\tau+x)d\tau}\, d\gamma\ , \quad (3.42)$$

keeping in mind that $e^{-\frac{i}{\hbar}\int_{-\infty}^{0} V(\gamma(\tau)+v\tau+x))d\tau}$ is not necessarely Fresnel integrable on $\mathcal{H}_-$, and that the integral in (3.42) is defined by the limit (3.41).

On the other hand (3.42) can also be defined without the limit procedure (3.41) in the case where $V(x)$ is a potential such that the perturbation $H = H_o + V$ is gentle. For gentle perturbations see for instance the references [45] and [46]. One has for example that if $V \in L_1(R^n) \cap L_\infty(R^n)$ then the perturbation $H = H_o + V$ is gentle and, for the case of R^3, one has the stronger result that if $V \in L_{3/2}(R^3)$ then the perturbation is gentle. [Ref [45] theorems 4, 5 and 6].

In the case when V is gentle and with small gentleness norm, i.e. if for instance $\|V\|_1$ and $\|V\|_\infty$ are bounded by a certain constant or, in the case of R^3 , if $\|V\|_{3/2}$ is bounded by a certain constant, then (3.42) may also be defined as follows, setting now again $m = \hbar = 1$:

$$\widetilde{\int_{\mathcal{H}_-}} e^{\frac{i}{2}\int_{-\infty}^{0}\left|\frac{d\gamma}{d\tau}\right|^2 d\tau}\ e^{-i\int_{-\infty}^{0} V(\gamma(\tau)+v\tau+x)}\ d\gamma$$

$$= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\widetilde{\int_{\mathcal{H}_-}} e^{\frac{i}{2}\int_{-\infty}^{0}\left|\frac{d\gamma}{d\tau}\right|^2 d\tau}\ \left(\int_{-\infty}^{0} V(\gamma(\tau)+v\tau+x)d\tau\right)^n d\gamma \qquad (3.43)$$

$$= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\widetilde{\int_{\mathcal{H}_-}} e^{\frac{i}{2}\int_{-\infty}^{0}\left|\frac{d\gamma}{d\tau}\right|^2 d\tau}\ V(\gamma(t_1)+vt_1+x)\ldots V(\gamma(t_n)+vt_n+x)d\gamma\, dt_1\ldots\ldots dt_n\ .$$

We have namely that $V(\gamma(t)+vt+x) \in \mathcal{F}(\mathcal{H}_-)$, hence the normalized integral in the last line in (3.43) is well defined.

In the same manner as earlier we may compute this normalized integral, and we get by the earlier computations the right hand side of (3.35), with $t = \infty$. Again by the earlier computations we see that this is the same series as the perturbation expansion of W_+ in powers of V, namely

$$\sum_{n=0}^{\infty} (-i)^n \int_{0 \le t_1 \le \dots \le t_n} \cdots \int V(t_1) \dots V(t_n) dt_1 \dots dt_n . \tag{3.44}$$

By the assumption that the perturbation is small and gentle we have that the integrals and the sum in (3.44) actually converge and the sum in (3.44) is equal to the wave operator W_+. Hence the integrals and the sum in (3.43) also converge and the sum is equal to $W_+(x,\beta)$ with $\beta = mv$.

Let us now take $t = 0$ and $s < 0$ in the formula (3.24), and substitute (3.27) for the expression (3.25). We then have

$$(e^{-isH_0} e^{isH} \varphi)(x) = \sum_{n=0}^{\infty} (-i)^n \int_{s < t_1 \le \dots \le t_n \le 0} \cdots \int e^{-\frac{i}{2}[\sum_{j,k=1}^{n} -t_j \vee t_k \alpha_j \alpha_k + 2\delta \sum_{j=1}^{n} t_j \alpha_j]} \cdot e^{i\delta x} d\nu(\beta) \prod_{i=1}^{n} (d\mu(\alpha_i) dt_i) . \tag{3.45}$$

Due to the symmetry of the integrand we get, after a substitution $t_i \to - t_i$, $i = 1, \dots, n$:

$$(e^{-isH_0} e^{isH} \varphi)(x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^{-s} \cdots \int_0^{-s} \int \cdots \int e^{-\frac{i}{2}[\sum_{j,k=1}^{n} t_j \wedge t_k \alpha_j \alpha_k - 2\delta_n \sum_{j=1}^{n} t_j \alpha_j]} \cdot e^{i\delta_n x} d\nu(\beta) \prod_{i=1}^{n} d\mu(\alpha_i) dt_1 , \tag{3.46}$$

with $\delta_n = \sum_{i=1}^{n} \alpha_i + \beta$.

For $s < 0$ we now define $W_s^*(\delta, y)$ by

$$\int W_s^*(\delta,y)\, e^{-i\delta y}\, \varphi(y)dy = \int (e^{-isH_o}\, e^{isH}\, \varphi)(x) e^{-i\delta x}\, dx \,. \tag{3.47}$$

So that, with ψ in $L_2(R^n)$ and

$$\begin{aligned}\psi(x) &= \int e^{i\delta x}\, d\sigma(\delta) \\ &= \int e^{i\delta x}\, \hat{\psi}(\delta) d\delta \,,\end{aligned} \tag{3.48}$$

we have

$$\iint W_s^*(\delta,y)\, e^{-i\delta y}\, \varphi(y)dy d\bar{\sigma}(\delta) = (\psi, e^{-isH_o}\, e^{isH}\, \varphi) \,. \tag{3.49}$$

From (3.46) we get

$$\begin{aligned}\int W_s^*(\delta,y) e^{-i\delta y}\, \varphi(y)dy d\bar{\sigma}(\delta) &= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_0^{-s}\!\!\cdots\!\int_0^{-s} \int\cdots\int \cdot \\ &\cdot e^{-\frac{i}{2}[\sum_{j,k=1}^{m} t_j \wedge t_k \alpha_j \alpha_k - 2(\beta+\sum_{j=1}^{m}\alpha_j)\sum_{j=1}^{m} t_j \alpha_j]} \\ &\cdot e^{i(\beta+\sum_{j=1}^{m}\alpha_j)x}\, \bar{\psi}(x) dx d\nu(\beta) \prod_{i=1}^{m} d\mu(\alpha_i) dt_i\end{aligned} \tag{3.50}$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_0^{-s}\!\!\cdots\!\int_0^{-s} \int\cdots\int e^{-\frac{i}{2}[\sum_{j,k=1}^{m} t_j \wedge t_k \alpha_j \alpha_k - 2(\beta+\sum_{j=1}^{m}\alpha_j)\sum_{j=1}^{m} t_j \alpha_j]}$$

$$\cdot (2\pi)^n\, \hat{\psi}(\beta + \sum_{j=1}^{m} \alpha_j) d\nu(\beta) \prod_{i=1}^{m} d\mu(\alpha_i) dt_i \,.$$

With the notation $\hat{\varphi}(\beta)d\beta = d\nu(\beta)$ and the substitution $\beta + \sum_{j=1}^{m} \alpha_j \to \delta$ we have that (3.50) is equal to

$$\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_0^{-s}\!\!\cdots\!\int_0^{-s} \int\cdots\int e^{-\frac{i}{2}[\sum_{j,k=1}^{m} t_j \wedge t_k \alpha_j \alpha_k - 2\delta\sum_{j=1}^{m} t_j \alpha_j]}$$

$$(2\pi)^n\, \hat{\psi}(\delta)\hat{\varphi}(\delta - \sum_{j=1}^{m}\alpha_j) d\delta \prod_{i=1}^{m} d\mu(\alpha_i) dt_i \,.$$

Using now the inverse Fourier transform

$$(2\pi)^n \hat{\varphi}(\delta) = \int \varphi(x) e^{-i\delta x}\, dx \ ,$$

we get (3.50) equal to

$$\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_0^{-s} \cdots \int_0^{-s} \int \cdots \int e^{-\frac{i}{2}[\sum_{j,k=1}^{m} t_j \wedge t_k \alpha_j \alpha_k - 2\delta \sum_{j=1}^{m} t_j \alpha_j]}$$

$$\cdot\ e^{ix \sum_{j=1}^{m} \alpha_j}\ e^{-i\delta x}\ \hat{\psi}(\delta)\varphi(x)\, dx\, d\delta \prod_{j=1}^{m} d\mu(\alpha_j)\, dt_j \ .$$

Hence we have proved the formula

$$W_s^*(\delta,x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^{-s} \cdots \int_0^{-s} \int \cdots \int e^{-\frac{i}{2}[\sum_{j,k=1}^{n} t_j \wedge t_k \alpha_j \alpha_k - 2\delta \sum_{j=1}^{n} t_j \alpha_j]}$$

$$\cdot\ e^{ix(\sum_{j=1}^{n} \alpha_j)} \prod_{j=1}^{n} d\mu(\alpha_j)\, dt_j \ . \qquad (3.51)$$

We introduce now the real separable Hilbert space $\mathcal{H}_+$ of continuous functions γ from $[0,\infty]$ to R^n such that $\gamma(0) = 0$ and $\frac{d\gamma}{d\tau}$ is in $L_2([0,\infty],R^n)$ with norm given by

$$|\gamma|^2 = \int_0^{\infty} |\frac{d\gamma}{d\tau}|^2 d\tau \ .$$

We verify easely that the function $s \wedge t$ plays the same role in $\mathcal{H}_+$ as the function $-s \vee t$ in $\mathcal{H}_-$, and thus by the same calculations as for $\mathcal{H}_-$ we get that $\int_0^{-s} V(\gamma(\tau) + \delta\tau + x)\, d\tau$ is in $\mathcal{F}(\mathcal{H}_+)$ and that

$$W_s^*(\delta,x) = \tilde{\int_{\mathcal{H}_+}} e^{\frac{i}{2}\int_0^{\infty}|\frac{d\gamma}{d\tau}|^2 d\tau}\ e^{-i\int_0^{-s} V(\gamma(\tau)+\delta\tau+x)\, d\tau}\, d\gamma \qquad (3.52)$$

If m or $\hbar$ are different from 1 we shall define the norm in $\mathcal{H}_+$ by

$$|\gamma|^2 = \frac{m}{\hbar}\int|\frac{d\gamma}{d\tau}|^2 d\tau$$

and we get the corresponding formula

$$W_s^*(mv,x) = \widetilde{\int_{\mathcal{H}_+}} e^{\frac{im}{2\hbar}\int_0^\infty|\frac{d\gamma}{d\tau}|^2 d\tau} \cdot e^{-\frac{i}{\hbar}\int_0^{-s} V(\gamma(\tau)+v\tau+x)d\tau} d\gamma \ . \qquad (3.53)$$

Let us now assume that the potential is so that the wave operators $W_\pm$ defined by (3.23) exist, then we get from (3.49) that

$$\int W_s^*(\delta,x)\, e^{-i\frac{\delta}{\hbar}x}\hat{\psi}(\delta)d\delta = (e^{-isH} e^{isH_0}\psi)(x) \qquad (3.54)$$

and the limit of (3.54) as $s \to -\infty$ exists in the strong L_2-sense and defines $W_-^*(\delta,x)$ by

$$\int W_-^*(\delta,x)e^{-i\frac{\delta}{\hbar}x}\hat{\psi}(\delta)d\delta = (W_-\psi)(x) \ . \qquad (3.55)$$

By the physical interpretation of the wave operator W_- we have from (3.55) that $W_-^*(\delta,x)$, as a function of δ for fixed x , is the asymptotic probability amplitude in momentum space as $t \to +\infty$ of a particle located at x for $t = 0$. In the same way as for $W_+(x,\beta)$ we may introduce the improper normalized integral as the limit as $s \to -\infty$ in the weak sense in $\delta = mv$ and x of the normalized integral (3.53) and then write

$$W_-^*(mv,x) = \widetilde{\int_{\mathcal{H}_+}} e^{\frac{im}{2\hbar}\int_0^\infty|\frac{d\gamma}{d\tau}|^2 dt} e^{-\frac{i}{\hbar}\int_0^\infty V(\gamma(\tau)+v\tau+x)d\tau} d\gamma \qquad (3.56)$$

or

$$W_-^*(mv,x) = \widetilde{\int_{\gamma(0)=x}} e^{\frac{im}{2\hbar}\int_0^\infty|\frac{d\gamma}{d\tau}|^2 d\tau} e^{-\frac{i}{\hbar}\int_0^\infty V(\gamma(\tau)+v\tau)d\tau} d\gamma \qquad (3.57)$$

Of course in the case of gentle perturbations (3.56) or (3.57) may also be defined by series expansion of the second term in the integral. Since the scattering operator S is defined by

$$S = W_-^* W_+ \tag{3.58}$$

we get that the scattering amplitude $S(\delta,\beta)$, which is simply the kernel of S in the momentum or Fourier transformed space, is given by the formula

$$S(\delta,\beta) = \int_{R^n} W_-^*(\delta,x) e^{\frac{i}{\hbar}(\beta-\delta)x} W_+(x,\beta)dx \ , \tag{3.59}$$

where we have taken $\hbar \neq 1$ and the integration is to be understood in the weak sense. (3.59) now gives a very interesting and surprising formula for the scattering amplitude

$$S(mv_+,mv_-) = \tilde{\int\limits_{\lim\limits_{t\to\pm\infty}\frac{\gamma(t)}{t}=v_\pm}} e^{\frac{i}{\hbar}(S(\gamma)-S_0(\gamma_0))} d\gamma \tag{3.60}$$

where $\gamma_0(\tau) = v_-\cdot\tau\wedge 0 + v_+\cdot\tau\vee 0 + y$ are the asymptotes of $\gamma(\tau)$, $S(\gamma)$ is the action along the path γ and $S_0(\gamma_0)$ is the free action along the asymptotic path γ_0 . (3.60) then expresses the quantum mechanical scattering amplitude as a normalized integral, over all paths with given asymptotic behaviour, of $\exp\{\frac{i}{\hbar}(S(\gamma)-S_0(\gamma_0))\}$, where $S(\gamma)-S_0(\gamma_0)$ is the difference of the action along γ and the free action along its asymptotes γ_0 . More precisely

$$S(\gamma)-S_0(\gamma_0) = \int_{-\infty}^{\infty}(\frac{m}{2}|\frac{d\gamma}{d\tau}|^2 - V(\gamma) - \frac{m}{2}|\frac{d\gamma_0}{d\tau}|^2)d\tau \tag{3.61}$$

and with $\gamma = \tilde{\gamma} + \gamma_0$ we get

$$S(\gamma) - S_o(\gamma_o) = \frac{m}{2}\int_{-\infty}^{\infty} \left|\frac{d\tilde{\gamma}}{d\tau}\right|^2 - \int_{-\infty}^{\infty} V(\tilde{\gamma} + \gamma_o) + mv_+ \int_0^{\infty} \frac{d\tilde{\gamma}}{d\tau}\, d\tau + mv_- \int_{-\infty}^{0} \frac{d\tilde{\gamma}}{d\tau}\, d\tau$$

i.e.

$$S(\gamma) - S_o(\gamma_o) = \frac{m}{2}\int_{-\infty}^{\infty} \left|\frac{d\tilde{\gamma}}{d\tau}\right|^2 - \int_{-\infty}^{\infty} V(\tilde{\gamma} + \gamma_o) - \tilde{\gamma}(0)(mv_+ - mv_-) \ . \tag{3.62}$$

Remark: Although $S(\gamma)$ and $S_o(\gamma_o)$ diverge, we see that $S(\gamma) - S_o(\gamma_o)$ is well defined whenever $\tilde{\gamma}$ is absolutely continuous with derivative in $L_2(R)$, if v_+ and v_- are different from zero and the potential V tends to zero faster than $|x|^{-1-\epsilon}$ for some positive ϵ. It is interesting to note that this are just the conditions that are needed for the existence of the scattering amplitude in quantum mechanics.

Let now $\tilde{\gamma}(0) = x$, we may then set

$$\tilde{\gamma}(\tau) = \begin{cases} \gamma_+(\tau) + x & \text{for } \tau \geq 0 \\ \gamma_-(\tau) + x & \text{for } \tau \leq 0 \ , \end{cases}$$

where $\gamma_\pm \in \mathcal{H}_\pm$. With this notation we get from (3.62) that

$$S(\gamma) - S_o(\gamma_o) = \frac{m}{2}\int_0^{\infty} \left|\frac{d\gamma_+}{d\tau}\right|^2 d\tau - \int_0^{\infty} V(\gamma_+ + v_+\tau + x)\,d\tau - x(mv_+ - mv_-)$$
$$+ \frac{m}{2}\int_{-\infty}^{0} \left|\frac{d\gamma_-}{d\tau}\right|^2 d\tau - \int_{-\infty}^{0} V(\gamma_- + v_-\tau + x)\,d\tau \ .$$

Hence, using the identity (3.63) we give a precise meaning to the **normalized** integral (3.60) by the following definition

$$\widetilde{\int\limits_{\lim_{t\to\pm\infty}\frac{\gamma(t)}{t}=v_\pm}} e^{\frac{i}{\hbar}(S(\gamma)-S_0(\gamma_0))} d\gamma$$

$$= \int_{R^n} dx\, e^{\frac{i}{\hbar}mx(v_+-v_-)} \widetilde{\int\limits_{\mathcal{H}_-\oplus\mathcal{H}_+}} e^{\frac{im}{2\hbar}\int_{-\infty}^{0}\left|\frac{d\gamma_-}{d\tau}\right|^2 d\tau + \frac{im}{2\hbar}\int_0^\infty \left|\frac{d\gamma_+}{d\tau}\right|^2 d\tau} \tag{3.64}$$

$$e^{-\frac{i}{\hbar}\int_{-\infty}^{0} V(\gamma_- + v_-\tau + x)d\tau} \cdot e^{-\frac{i}{\hbar}\int_0^\infty V(\gamma_+ + v_+\tau + x)d\tau} d\gamma ,$$

where the normalized integral above is the normalized integral on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\gamma = \gamma_+ \oplus \gamma_-$.

From (3.64), (3.59), (3.56) and (3.42) we have proved the formula (3.60) as defined by (3.64). We formulate now these results in the following theorem.

Theorem 3.2.

Let the potential V be the Fourier transform of a bounded complex measure and also in a class of potentials such that the wave operators (3.23) exist. Then the wave amplitudes $W_+(x,\beta)$ and $W_-^*(\delta,x)$ defined by (3.39) and (3.55) are given by the following improper normalized integrals

$$W_+(x,mv) = \widetilde{\int\limits_{\mathcal{H}_-}} e^{\frac{im}{2\hbar}\int_{-\infty}^{0}\left|\frac{d\gamma}{d\tau}\right|^2 d\tau} e^{-\frac{i}{\hbar}\int_{-\infty}^{0} V(\gamma(\tau)+v\tau+x)d\tau} d\gamma$$

and

$$W_-^*(mv,x) = \widetilde{\int\limits_{\mathcal{H}_+}} e^{\frac{im}{2\hbar}\int_0^\infty\left|\frac{d\gamma}{d\tau}\right|^2 d\tau} e^{-\frac{i}{\hbar}\int_0^\infty V(\gamma(\tau)+v\tau+x)d\tau} d\gamma ,$$

where the improper normalized integrals are defined as the limits of the corresponding ordinary normalized integrals with the integrals over the half lines of the function $V(\gamma(\tau)+v\tau+x)$ substituted by the integrals over finite time intervals of the same function. $\mathcal{H}_+$ is the real separable Hilbert space of absolutely continuous functions γ from $[0,\infty]$ to R^n such that $\gamma(0)=0$ and $\frac{d\gamma}{d\tau}$ is in $L_2(R)$, with norm $|\gamma|_+^2 = \frac{m}{\hbar}\int_0^\infty |\frac{d\gamma}{d\tau}|^2 d\tau$.

$\mathcal{H}_-$ is the corresponding space of functions on $[-\infty,0]$. Moreover the scattering amplitude $S(\alpha,\beta)$, defined as the Fourier transform of the kernel of the scattering operator S, is given by the formula

$$S(mv_+,mv_-) = \widetilde{\int\limits_{\lim\limits_{t\to\pm\infty}\frac{\gamma(t)}{t}=v_\pm}} e^{\frac{i}{\hbar}(S(\gamma)-S_0(\gamma_0))} d\gamma ,$$

where

$$S(\gamma) - S_0(\gamma_0) = \int_{-\infty}^{\infty} (\frac{m}{2}|\frac{d\gamma}{d\tau}|^2 - V(\gamma) - \frac{m}{2}|\frac{d\gamma_0}{d\tau}|^2)d\gamma$$

with $\gamma_0(\tau) = v_-\cdot\tau\wedge 0 + v_+\cdot\tau\vee 0 + y$, and the normalized integral above is defined by (3.64). □

4. The Fresnel integral relative to a non singular quadratic form.

In theorem 3.1 we obtained the solution $\psi(x,t)$ of the Schrödinger equation with initial values $\varphi(x)$ and potential $V(x)$ in the form

$$\psi(x,t) = \widetilde{\int_{\gamma(t)=x}} e^{\frac{im}{2\hbar}\int_0^t \left|\frac{d\gamma}{d\tau}\right|^2 d\tau \;-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau} \varphi(\gamma(0))d\gamma \qquad (4.1)$$

$$\underset{=}{\mathrm{Def}} \quad \widetilde{\int_{\mathcal{H}}} e^{\frac{im}{2\hbar}\int_0^t \left|\frac{d\gamma}{d\tau}\right|^2 d\tau} \cdot e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau)+x)d\tau} \varphi(\gamma(0)+x)d\gamma \ ,$$

where $\mathcal{H}$ was the real Hilbert space of continuous paths γ such that $\gamma(t) = 0$ and with norm square given by $\frac{m}{\hbar}\int_0^t \left|\frac{d\gamma}{d\tau}\right|^2 d\tau$, if both V and φ are Fourier transforms of bounded complex measures. If we are, and we shall be, interested in the anharmonic oscillator, then we must deal with potentials of the form

$$V'(x) = \tfrac{1}{2}xA^2x + V(x) \ , \qquad (4.2)$$

where xA^2x is a strictly positive definite form on R^n , corresponding to the strictly positive definite symmetric linear transformation A^2 on R^n , and $V(x)$ is a nice function, which we shall take to be in the class of Fourier transforms of bounded complex measures. For such potentials, of the form (4.2), we can not prove formula (4.1) in the same way as in the previous section, because if we substitute V' for V in (4.1) we do not get a Fresnel integrable function on $\mathcal{H}$ and the formula therefore does not make sense as it stands. On the other hand, we may write (4.1) with V' instead of V in the following manner

$$\widetilde{\int_{\gamma(t)=x}} e^{\frac{im}{2\hbar}\int_0^t \dot{\gamma}(\tau)^2 d\tau - \frac{i}{2\hbar}\int_0^t \gamma(\tau)A^2\gamma(\tau)d\tau} \, e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau} \varphi(\gamma(0))d\gamma \ , \qquad (4.3)$$

where $\frac{d\gamma}{d\tau}(\tau) = \dot{\gamma}(\tau)$.

For small values of t

$$\frac{m}{\hbar}\int_0^t \dot{\gamma}(\tau)^2 d\tau - \frac{1}{\hbar}\int_0^t \gamma(\tau)A^2\gamma(\tau)d\tau \tag{4.4}$$

is namely a strictly positive definite quadratic form on the space of continuous paths such that $\gamma(t) = 0$. Hence we may introduce the real Hilbert space $\mathcal{H}'$ of continuous functions from $[0,t]$ to R^n such that (4.4) is bounded with (4.4) as the norm square, and define (4.3) by

$$\tilde{\int_{\mathcal{H}'}} e^{\frac{i}{2}[\frac{m}{\hbar}\int_0^t \dot{\gamma}(\tau)^2 d\tau - \frac{1}{\hbar}\int_0^t (\gamma(\tau)+x)A^2(\gamma(\tau)+x)d\tau]} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau)+x)d\tau} \varphi(\gamma(0)+x)d\gamma \ , \tag{4.5}$$

where the normalized integral $\tilde{\int_{\mathcal{H}'}}$ is the one defined in section 2. One verifies then easely that

$$e^{-\frac{i}{\hbar} xA^2 \int_0^t \gamma(\tau)d\tau} e^{-i/\hbar \int_0^t V(\gamma(\tau)+x)} \varphi(\gamma(0)+x)\cdot e^{-\frac{it}{2\hbar} x A^2 x} \tag{4.6}$$

is Fresnel integrable on $\mathcal{H}'$ and that, up to a constant, the Fresnel integral (4.5) gives the solution of the corresponding Schrödinger equation at time t . The constant, which depends only on t, m and A^2 , comes from the fact that the normalized integral is defined by a normalization given by the inner product in the Hilbert space. In this way we can thus prove an analogue of the Feynman-Ito formula also for the anharmonic oscillator. But this formula would then only hold for small values of t . This is of course rather unsatisfactory, and we shall therefore not give the detailed proof here.

If on the other hand we want to make sense out of (4.3) not only for small t, we must define the Fresnel integral also for the case where the quadratic form in question is not necessarely positive definite any longer. We shall see that this is possible and in this way make sense of (4.3) not only for small values of t. To define this extension of the Fresnel integral we shall introduce a densely defined symmetric operator B in the separable real Hilbert space $\mathcal{H}$ of section 2, and the quadratic form is then given by

$$(x, Bx) \tag{4.7}$$

for $x \in D(B) \subset \mathcal{H}$. It is also necessary to assume that B is non degenerate in some suitable sense, and we shall here assume that B is non degenerate in the following sense. There exists a dense subspace D of $\mathcal{H}$ such that D contains the range of B and there exists a symmetric bilinear form $\Delta(x,y)$ defined on $D \times D$ such that $\operatorname{Im} \Delta(x,x) \leq 0$ and

$$\Delta(x,By) = (x,y) \qquad 4.8)$$

for all $x \in D$ and all y in the domain $D(B)$ of B. Δ is in the above sense an inverse form of the form (4.7), and in the case where $\mathcal{H}$ is finite dimensional the existence of Δ is equivalent to B having an inverse and Δ is in that case given by the inverse matrix of B. We shall further assume that D is a separable Banach space with norm $\|x\|$ which is stronger than the norm $|x|$ in $\mathcal{H}$, i.e.

$$|x| \leq a\|x\| \tag{4.9}$$

for all $x \in D$, and we shall assume that the form $\Delta(x,y)$ is a continuous symmetric bilinear form on D. From (4.9) and the self duality of $\mathcal{H}$ we get the natural imbedding

$$D \subset \mathcal{H} \subset D^* , \tag{4.10}$$

where D^* is the dual space of D. The imbedding $\mathcal{H} \subset D^*$ is just the restriction mapping i.e. by restricting a continuous linear function on $\mathcal{H}$ to D we get, by (4.9), a continuous linear mapping on D. It also follows from (4.9) that $\|x\|_* \leq a|x|$, where $\|x\|_*$ is the norm in D^*. Hence all the injections in (4.10) are continuous. In the general case Δ is not uniquely given by B. However if B is non degenerate in the stronger sense that it has a bounded continuous inverse B^{-1} on $\mathcal{H}$, then it follows easely that $D = \mathcal{H}$, since D contains the range of B, and that $\Delta(x,y) = (x,B^{-1}y)$. Hence Δ is in this case unique and real.

Since $\Delta(x,y)$ is continuous on $D \times D$ we have that, for fixed x, $\Delta(x,y)$ is a continuous complex linear functional on D, hence in D^*. This gives then a mapping of D into D^* which is, by (4.8), a left inverse of B, considered as a map from $D(B) \subset D^*$ into D.

We now define the Fresnel integral with respect to the form Δ.

<u>Definition 4.0</u>

Let D be a real separable Banach space with norm $\| \ \|$ and let $\mathcal{H}$ be a real separable Hilbert space with inner product $(,)$ and norm $| \ |$, such that D is densely contained in $\mathcal{H}$ and the norm in D is stronger than the norm in $\mathcal{H}$. Let B be a densely defined symmetric operator on $\mathcal{H}$ such that the range of B is contained in D and let $\Delta(x,y)$ be a symmetric and continuous bilinear form on $D \times D$ such that $\operatorname{Im} \Delta(x,x) \leq 0$, and $\Delta(x,By) = (x,y)$ for all $x \in D$ and $y \in D(B)$. The space $\mathcal{F}(D^*)$ of Fresnel integrable functions on D^* is the space of Fourier transforms of bounded complex measures on D. For any $f \in \mathcal{F}(D^*)$ we get, by the inclusion $\mathcal{H} \subset D^*$, that

$$f(x) = \int_D e^{i(x,y)} \, d\mu(y) \tag{4.11}$$

for $x \in \mathcal{H}$. We now define, for any $f \in \mathcal{F}(D^*)$, the Fresnel integral with respect to Δ by

$$\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(x,Bx)} f(x)dx = \int_D e^{-\frac{i}{2}\Delta(x,x)} \, d\mu(x) \,. \tag{4.12}$$

The integral on the right hand side is well defined since $\operatorname{Im} \Delta(x,x) \leq 0$ and $\Delta(x,x)$ is continuous on D. We shall also call $\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(x,Bx)} f(x)dx$ the integral normalized with respect to Δ of the function $e^{\frac{i}{2}(x,Bx)} f(x)$, and also use the notation $\mathcal{F}_{\Delta}(f)$ for this integral.

We have now the following proposition.

<u>Proposition 4.1</u>

The space of Fresnel integrable functions $\mathcal{F}(D^*)$ is a Banach-function-algebra in the norm $\|f\|_0 = \|\mu\|$ and $\mathcal{F}(D^*) \subseteq \mathcal{F}(\mathcal{H})$. $\mathcal{F}_{\Delta}(f)$ is a bounded continuous linear functional on $\mathcal{F}(D^*)$ such that $|\mathcal{F}_{\Delta}(f)| \leq \|f\|_0$ and normalized such that $\mathcal{F}_{\Delta}(1) = 1$. The condition $\Delta(x,By) = (x,y)$ for $x \in D$ and $y \in D(B)$ implies that B^{-1} is well defined and $\Delta(x,y) = (x,B^{-1}y)$ for $x \in D$ and y in the range of B. Hence, if the range of B is dense in D, $\Delta(x,y)$ is uniquely given by B and is the continuous extension of the form $(x,B^{-1}y)$. If $B \geq a1$ with $a > 0$ and the range of B dense in D , then $\mathcal{F}(D^*) \subseteq \mathcal{F}(\mathcal{H}_B)$ and

$$\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(x,Bx)} f(x)dx = \int_{\mathcal{H}_B}^{\sim} e^{\frac{i}{2}|x|_B^2} f(x)dx ,$$

with $|x|_B^2 = (x,Bx)$ and where $\mathcal{H}_B$ is the closure of $D(B)$ in the norm $|\ |_B$.

Proof: Let $f \in \mathcal{F}(D^*)$, then, for $x \in \mathcal{H}$,

$$f(x) = \int_D e^{i(x,y)} d\mu(y) .$$

Since the D-norm is stronger than the $\mathcal{H}$-norm, we have that the $\mathcal{H}$-norm $|x|$ is a continuous function on D , from which it follows that any $\mathcal{H}$-continuous function is also D-continuous, so the restriction of the integral with respect to μ from $C(D)$ to $C(\mathcal{H})$ gives a measure on $\mathcal{H}$, which we shall denote by $\mu_{\mathcal{H}}$. Since $e^{i(x,y)}$ for $x \in \mathcal{H}$ is in $C(\mathcal{H})$, we therefore have that, by the definition of $\mu_{\mathcal{H}}$,

$$f(x) = \int_{\mathcal{H}} e^{i(x,y)} d\mu_{\mathcal{H}}(y) , \tag{4.13}$$

hence that $f \in \mathcal{F}(\mathcal{H})$. That $\|\mu_{\mathcal{H}}\| = \|\mu\|$ is obvious, so that $\mathcal{F}(D^*)$ is a Banach subspace of $\mathcal{F}(\mathcal{H})$. That it is also a Banach algebra follows as in proposition 2.2 from the fact that D is a separable metric group. The bound $|\mathcal{F}_\Delta(f)| \leq \|f\|_0$ follows from the fact that $\mathrm{Im}\, \Delta(x,x) \leq 0$, and $\mathcal{F}_\Delta(1) = 1$ is obvious from the definition (4.12). From

$$\Delta(x,By) = (x,y) , \tag{4.14}$$

for $x \in D$ and any $y \in D(B)$, we get that $By = 0$ implies that $y = 0$, since D is dense in $\mathcal{H}$. Hence B^{-1} is well defined with domain equal to the range of B. Hence

$$\Delta(x,y) = (x,B^{-1}y) \tag{4.15}$$

for $x \in D$ and y in the range of B. So that, if the range of B is dense in D, then $\Delta(x,y)$ is uniquely given by (4.15) and therefore also real.

If $B \geq a1$ with $a > 0$ and the range cf B is dense in D, then, for $x \in R(B)$, the range of B,

$$|B^{-1}x|_B^2 = (x,B^{-1}x) = \Delta(x,x) , \tag{4.16}$$

which is bounded and continuous in the D-norm. Hence B^{-1} maps $R(B)$ into $\mathcal{H}_B$, continuously in the D-norm on $R(B)$. Since $R(B)$ is dense in D it has a unique continuous extension, which we shall also denote by B^{-1}, such that B^{-1} maps D into $\mathcal{H}_B$ boundedly. To prove the identity in the proposition we first prove that $\mathcal{F}(D^*) \subseteq \mathcal{F}(\mathcal{H}_B)$. Let $f \in \mathcal{F}(D^*)$, then for $x \in \mathcal{H}$ we have

$$f(x) = \int_D e^{i(x,y)}\, d\mu(y) . \tag{4.17}$$

Let $g \in C(\mathcal{H}_B)$, we define μ_B by

$$\int_{\mathcal{H}_B} g(x)d\mu_B(x) = \int_D g(B^{-1}x)d\mu(x) . \tag{4.18}$$

$g(B^{-1}x) \in C(D)$ since $B^{-1}: D \to \mathcal{H}_B$ continuously. If $x \in D(B)$ then by (4.17) and (4.18) we have

$$f(x) = \int_D e^{i(Bx,B^{-1}y)}\, d\mu(y)$$

$$= \int_{\mathcal{H}_B} e^{i(Bx,y)}\, d\mu_B(y) ,$$

so that

$$f(x) = \int_{\mathcal{H}_B} e^{i(x,y)_B}\, d\mu_B(y) . \tag{4.19}$$

Now (4.19) holds for all $x \in D(B)$ and $D(B)$ is by definition dense in $\mathcal{H}_B$ in the $\mathcal{H}_B$-norm. On the other hand the right hand side of (4.19) is obviously uniformly continuous in the $\mathcal{H}_B$-norm. But, from (4.17), $f(x)$, for $x \in \mathcal{H}$, is uniformly continuous in the $\mathcal{H}$-norm, which is weaker than the $\mathcal{H}_B$-norm. Hence by unique extension of uniformly continuous functions defined on dense subspaces, (4.19) must hold for all x in $\mathcal{H}_B$. This then proves that, in this case, $\mathcal{F}(D^*) \subset \mathcal{F}(\mathcal{H}_B)$, and by (4.19) we have

$$\tilde{\int_{\mathcal{H}_B}} e^{\frac{i}{2}|x|_B^2} f(x)dx = \int_{\mathcal{H}_B} e^{-\frac{i}{2}|x|_B^2} d\mu_B(x)$$

$$= \int_D e^{-\frac{i}{2}|B^{-1}x|_B^2} d\mu(x) .$$

On the other hand we have (4.16) for all $x \in R(B)$, but by the continuity of $B^{-1}: D \to \mathcal{H}_B$ and the continuity of $\Delta(x,x)$ in the D-norm we have that $|B^{-1}x|_B = \Delta(x,x)$ for all x in D, hence

$$\tilde{\int_{\mathcal{H}_B}} e^{\frac{i}{2}|x|_B^2} f(x)dx = \int_D e^{-\frac{i}{2}\Delta(x,x)} d\mu(x) ,$$

which by (4.17) proves the identity in the proposition. □

<u>Proposition 4.2</u>

The integral normalized with respect to Δ is invariant under translations by vectors in the domain of B, i.e., for $y \in D(B)$,

$$\overset{\Delta}{\int_{\mathcal{H}}} e^{\frac{i}{2}(x+y,B(x+y))} f(x+y)dx = \overset{\Delta}{\int_{\mathcal{H}}} e^{\frac{i}{2}(x,Bx)} f(x)dx .$$

If $\mathcal{H}$ is finite dimensional then B^{-1} is bounded and Δ is is uniquely given by $\Delta(x,y) = (x,B^{-1}y)$ for all x and y, and with $\mathcal{H} = R^n$

$$\int\limits_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(x,Bx)} f(x)dx = |-2\pi iB|^{-\frac{1}{2}} \int\limits_{R^n} e^{\frac{i}{2}(x,Bx)} f(x)dx$$

for any $f \in \mathcal{S}(R^n)$, where $|-2\pi iB|$ is the determinant of the transformation $-2\pi iB$ in R^n, and the integral on the right hand side is the Lebesgue integral.

If $\mathcal{H}$ is infinite dimensional and B^{-1} is bounded and everywhere defined, then $D = \mathcal{H}$, their norms are equivalent and Δ is uniquely given by $\Delta(x,y) = (x,B^{-1}y)$ for all x and y. Moreover B is self adjoint and $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, with $B_+ \ominus B_-$, where $B_{\pm} \geq a1$ for some $a > 0$, and

$$\int\limits_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(x,Bx)} f(x)dx = \overline{\int\limits_{\mathcal{H}_{B_+}}^{\sim} e^{\frac{i}{2}|x_1|^2_{B_+}} \left[\int\limits_{\mathcal{H}_{B_-}}^{\sim} e^{\frac{i}{2}|x_2|^2_{B_-}} f(x_1,x_2)dx_2\right]dx_1} ,$$

where $f(x_1,x_2) = f(x_1 \oplus x_2)$, $\mathcal{H}_{B_+}$ is the closure of $D(B_{\pm}) \subset \mathcal{H}_+$ in the norm $|x|_{B_{\pm}}^2 = (x,B_{\pm}x)$, and the integrals on the right hand side are ordinary normalized integrals as defined in section 2.

Proof: Let $y \in D(B)$, then

$$e^{\frac{i}{2}(x+y,B(x+y))} f(x+y) = e^{\frac{i}{2}(y,By)} e^{\frac{i}{2}(x,Bx)} e^{i(x,By)} f(x+y), \tag{4.20}$$

so we shall first prove that $e^{i(x,By)} f(x+y) \in \mathcal{F}(D^*)$. We have

$$\begin{aligned} e^{i(x,By)} f(x+y) &= e^{i(x,By)} \int\limits_D e^{i(x+y,z)} d\mu(z) \\ &= \int\limits_D e^{i(x,z)} e^{i(x,By)} e^{i(y,z)} d\mu(z) \\ &= \int\limits_D e^{i(x,z+By)} e^{i(y,z)} d\mu(z) , \end{aligned}$$

and since $By \in D$, so that $z \to z - By$ is a continuous transformation in D, we get

$$e^{i(x,By)} f(x+y) = \int_D e^{i(x,z)} e^{i(y,z-By)} d\mu(z-By) \ , \qquad (4.21)$$

which is obviously in $\mathcal{F}(D^*)$. Hence by (4.20), (4.21) and the definition of the integral normalized with respect to Δ we get

$$\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(x+y,B(x+y))} f(x+y)dx = e^{\frac{i}{2}(y,By)} \int_D e^{-\frac{i}{2}\Delta(x,x)} e^{i(y,x-By)} d\mu(x-By),$$

which by the definition of the measure $d\mu(x-By)$ is

$$e^{\frac{i}{2}(y,By)} \int_D e^{-\frac{i}{2}\Delta(x+By,x+By)} e^{i(y,x)} d\mu(x)$$

$$= e^{\frac{i}{2}(y,By)} e^{-\frac{i}{2}\Delta(By,By)} \int_D e^{-\frac{i}{2}\Delta(x,x)} e^{-i\Delta(x,By)} e^{i(y,x)} d\mu(x)$$

and, using now that for $y \in D(B)$ and $x \in D$ we have $\Delta(x,By) = (x,y)$, we get

$$\int e^{-\frac{i}{2}\Delta(x,x)} d\mu(x) \ .$$

This proves the translation invariance. We have proved previously that $By = 0 \Rightarrow y = 0$, which in the finite dimensional case implies that B is onto and that B^{-1} is bounded. In this case, since D is dense in $\mathcal{H}$, we have that $D = \mathcal{H}$ and therefore that $\Delta(x,y) = (x,B^{-1}y)$ for all x and y . Let now $f(x) \in \mathcal{S}(R^n)$, $R^n = \mathcal{H}$, then

$$\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(x,Bx)} f(x)dx \overset{\text{Def}}{=} \int e^{-\frac{i}{2}(x,B^{-1}x)} \hat{f}(x)dx \qquad (4.22)$$

with

$$f(x) = \int e^{i(x,y)} \hat{f}(y)dy \ .$$

On the other hand one verifies easely that, for $f \in \mathcal{S}(R^n)$,

$$\int e^{-\frac{i}{2}(x,B^{-1}x)} \hat{f}(x)dx = |-2\pi iB|^{-\frac{1}{2}} \int e^{\frac{i}{2}(x,Bx)} f(x)dx . \tag{4.23}$$

This proves the second part of the proposition. Let now $\mathcal{H}$ be infinite dimensional and B^{-1} bounded and everywhere defined. Then the range of B is $\mathcal{H}$, hence $D = \mathcal{H}$, so that the D-norm $\|x\|$ is a norm on $\mathcal{H}$ which is everywhere defined, hence by the general theory of functional analysis it is bounded with respect to the norm in $\mathcal{H}$. Therefore the D-norm and $\mathcal{H}$ norm are equivalent. Moreover since B is the inverse of a bounded symmetric operator, it is self adjoint and let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be the spectral decomposition of B in the subspaces where B is positive and B is negative. This decomposition is unique since we already know that zero is not an eigenvalue of B . Let $\pm B_\pm$ be the restrictions of B to $\mathcal{H}_\pm$. By the spectral representation theorem for self adjoint operators it follows from B^{-1} bounded that $B_\pm \geq a 1$ for some $a > 0$. Let now $f \in \mathcal{F}(D^*)$, then since $D = \mathcal{H}$ and D and $\mathcal{H}$ are equivalent as metric spaces and the fact that $\mathcal{H}$ and $\mathcal{H}_+ \times \mathcal{H}_-$ are equivalent as metric spaces, we get that any measure on D may be regarded as a measure on $\mathcal{H}_+ \times \mathcal{H}_-$ and therefore, for $x = x_1 \oplus x_2 \in \mathcal{H}$:

$$f(x) = \int_{\mathcal{H}_+ \times \mathcal{H}_-} e^{i(x_1,y_1)} e^{i(x_2,y_2)} d\mu(y_1,y_2) . \tag{4.24}$$

Consider now the measure μ_{x_1} on $\mathcal{H}_-$ defined by

$$\int_{\mathcal{H}_-} g(y_2)d\mu_{x_1}(y_2) = \int_{\mathcal{H}_+ \times \mathcal{H}_-} g(-y_2)\, e^{-i(x_1,y_1)} \overline{d\mu(y_1,y_2)} , \tag{4.25}$$

for any $g \in C(\mathcal{H}_-)$. Then

$$\overline{f(x_1,x_2)} = \int_{\mathcal{H}_-} e^{i(x_2,y_2)} \, d\mu_{x_1}(y_2) \,. \tag{4.26}$$

Let now $\mathcal{H}_{B_\pm}$ be the completion of $D(B) \cap \mathcal{H}_\pm$ in the norm $|x|_{B_\pm}^2 = (x,x)_{B_\pm} = (x, B_\pm x)$. Then for $x_2 \in D(B) \cap \mathcal{H}_-$ we have

$$\overline{f(x_1,x_2)} = \int_{\mathcal{H}_-} e^{i(x_2,B^{-1}y_2)_{B_-}} \, d\mu_{x_1}(y_2)$$

i.e.

$$\overline{f(x_1,x_2)} = \int_{\mathcal{H}_{B_-}} e^{i(x_2,y_2)_{B_-}} \, d\mu_{x_1}^{B_-}(y_2) \,, \tag{4.27}$$

where $d\mu_{x_1}^{B_-}$ is the measure on $\mathcal{H}_{B_-}$ defined by

$$\int_{\mathcal{H}_{B_-}} g(y) d\mu_{x_1}^{B_-}(y) = \int_{\mathcal{H}_-} g(B^{-1}y) d\mu_{x_1}(y) \,, \tag{4.28}$$

for any $g \in C(\mathcal{H}_{B_-})$. (4.28) defines a measure on $\mathcal{H}_{B_-}$, since

$$|B_-^{-1}y|_{B_-}^2 = (y, B_-^{-1}y) \le a\|y\|^2 \,, \tag{4.29}$$

so that B_-^{-1} maps $\mathcal{H}_-$ into $\mathcal{H}_{B_-}$ continuously. By (4.27), for fixed $x_1, \overline{f(x_1,x_2)} \in \mathcal{F}(\mathcal{H}_{B_-})$. Hence we may compute the inner integral in proposition 4.2, and we get for fixed x_1

$$\begin{aligned} g(x_1) &= \tilde{\int}_{\mathcal{H}_{B_-}} e^{\frac{i}{2}|x_2|_{B_-}^2} \, \overline{f(x_1,x_2)} dx_2 = \int_{\mathcal{H}_{B_-}} e^{-\frac{i}{2}|x_2|_{B_-}^2} \, d\mu_{x_1}^{B_-}(x_2) \\ &= \int_{\mathcal{H}_-} e^{-\frac{i}{2}|B_-^{-1}x_2|_{B_-}^2} \, d\mu_{x_1}(x_2) \end{aligned} \tag{4.30}$$

$$= \int_{\mathcal{H}_-} e^{-\frac{i}{2}(x_2, B_-^{-1} x_2)} \, d\mu_{x_1}(x_2) ,$$

which by the definition of $d\mu_{x_1}$ is equal to

$$\int_{\mathcal{H}_+ \times \mathcal{H}_-} e^{-\frac{i}{2}(y_2, B_-^{-1} y_2)} e^{-i(x_1, y_1)} \, \overline{d\mu(y_1, y_2)} .$$

Hence

$$\overline{g(x_1)} = \int_{\mathcal{H}_+ \times \mathcal{H}_-} e^{i(x_1, y_1)} e^{\frac{i}{2}(y_2, B_-^{-1} y_2)} \, d\mu(y_1, y_2) . \qquad (4.31)$$

Define now the measure $d\nu(y_1)$ on $\mathcal{H}_+$ by

$$\int_{\mathcal{H}_+} h(y_1) d\nu(y_1) = \int_{\mathcal{H}_+ \times \mathcal{H}_-} h(y_1) e^{\frac{1}{2}(y_2, B_-^{-1} y_2)} d\mu(y_1, y_2) , \qquad (4.32)$$

then

$$\overline{g(x_1)} = \int_{\mathcal{H}_+} e^{i(x_1, y_1)} \, d\nu(y_1)$$

$$= \int_{\mathcal{H}_+} e^{i(x_1, B_+^{-1} y_1)_{B_+}} \, d\nu(y_1) .$$

Hence

$$\overline{g(x_1)} = \int_{\mathcal{H}_{B_+}} e^{i(x_1, y_1)} \, d\nu^{B_+}(y_1) ,$$

so that $\bar{g} \in \mathcal{F}(\mathcal{H}_{B_+})$ and we may therefore compute the outer integral in the proposition and we get

$$\tilde{\int}_{\mathcal{H}_{B_+}} e^{\frac{i}{2}|x_1|^2_{B_+}} \overline{g(x_1)} \, dx_1 = \int_{\mathcal{H}_{B_+}} e^{-\frac{i}{2}|y_1|^2_{B_+}} d\nu^{B_+}(y_1)$$

$$= \int_{\mathcal{H}_+} e^{-\frac{i}{2}|B_+^{-1} y_1|^2_{B_+}} \, d\nu(y_1)$$

$$= \int_{\mathcal{H}_+} e^{-\frac{i}{2}(y_1, B_+^{-1} y_1)} \, d\nu(y_1) \; .$$

By the definition (4.32) of $d\nu(y_1)$ we get this equal to

$$\int_{\mathcal{H}_+ \times \mathcal{H}_-} e^{-\frac{i}{2}(y_1, B_+^{-1} y_1)} \, e^{\frac{i}{2}(y_2, B_-^{-1} y_2)} \, d\mu(y_1, y_2)$$

$$= \int_{\mathcal{H}} e^{-\frac{i}{2}(y, B^{-1} y)} \, d\mu(y) = \int_{D} e^{-\frac{i}{2}\Delta(y,y)} \, d\mu(y) \; ,$$

which by definition is the left hand side of the last equality in the proposition. This then completes the proof of this proposition.

□

The case where B is a bounded symmetric operator with $D(B) = R(B) = \mathcal{H}$, such that B^{-1} is bounded, deserves special attention. We have seen that in this case the space D must be equal to $\mathcal{H}$ and the form $\Delta(x,y)$ is unique and equal to $(x, B^{-1} y)$. Since Δ is unique we may drop it in the notation of the integral normalized with respect to Δ and we shall simply write

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\langle x,x\rangle} f(x)dx \overset{\text{Def}}{=} \overset{\Delta}{\int_{\mathcal{H}}} e^{\frac{i}{2}(x,Bx)} f(x)dx \; , \tag{4.33}$$

where $\langle x,y\rangle = (x,By)$ and $\Delta(x,y) = (x,B^{-1}y)$, in the case B and B^{-1} are both bounded with domains equal to $\mathcal{H}$. In this case we have, for any function $f \in \mathcal{F}(\mathcal{H})$, so that

$$f(x) = \int_{\mathcal{H}} e^{i(x,\alpha)} \, d\mu(\alpha) \; , \tag{4.34}$$

the representation

$$f(x) = \int_{\mathcal{H}} e^{i\langle x,\alpha\rangle} \, d\nu(\alpha) \; , \tag{4.35}$$

if we take ν to be the measure defined by

$$\int_{\mathcal{H}} h(\alpha)d\nu(\alpha) = \int_{\mathcal{H}} h(B^{-1}\alpha)d\mu(\alpha) .$$

It follows now from (4.33) that

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\langle x,x\rangle} f(x)dx = \int_{\mathcal{H}} e^{-\frac{i}{2}\langle\alpha,\alpha\rangle} d\nu(\alpha) . \tag{4.36}$$

From this we can see that it is natural to generalize the normalized integral on a Hilbert space treated in section 2 to the following situation. Let E be a real separable Banach space on which we have a non degenerate bounded symmetric bilinear form $\langle x,y\rangle$, where non degenerate simply means that the continuous mapping from E into E^*, the dual of E, given by the form $\langle x,y\rangle$, is one to one. Let $\mathcal{F}(E,\langle\ \rangle)$ be the Banach space of continuous functions on E of the form

$$f(x) = \int_E e^{i\langle x,\alpha\rangle} d\mu(\alpha) , \tag{4.37}$$

where μ is a bounded complex measure on E with norm $\|f\|_0 = \|\mu\|$. Since E is a separable metric group it easely follows as in section 2 that $\mathcal{F}(E,\langle\ \rangle)$ is a Banach algebra. We now define the normalized integral on E, equipped with the non degenerate form $\langle\ \rangle$, by

$$\tilde{\int_E} e^{\frac{i}{2}\langle x,x\rangle} f(x)dx = \int_E e^{-\frac{i}{2}\langle\alpha,\alpha\rangle} d\mu(\alpha) , \tag{4.38}$$

and from (4.36) we have that in the case where E is a separable Hilbert space and $\langle x,y\rangle = (x,By)$, where B and B^{-1} are both bounded and everywhere defined, the normalized integral (4.38) is the same as the integral normalized with respect to the form

$(x,B^{-1}x)$. It follows easely as in section 2 that (4.38) is a bounded continuous linear functional on $\mathcal{F}(E,\langle\ \rangle)$.

Proposition 4.2.

Let E_1 and E_2 be two real separable Banach spaces and let $\langle x_1,x_2\rangle$ be a non degenerate bounded symmetric bilinear form on E_1 . Let T be a bounded one-to-one mapping of E_2 into E_1 with a bounded inverse. Then $\langle Ty_1,Ty_2\rangle$ is a non degenerate bounded symmetric bilinear form on E_2. Moreover if $f \in \mathcal{F}(E_1,\langle,\rangle)$ then $f(Ty)$ is in $\mathcal{F}(E_2,\langle T\cdot,T\cdot\rangle)$ and

$$\tilde{\int_{E_1}} e^{\frac{i}{2}\langle x,x\rangle} f(x)dx = \tilde{\int_{E_2}} e^{\frac{i}{2}\langle Ty,Ty\rangle} f(Ty)dy \ .$$

Proof: The non degeneracy of $\langle Ty_1,Ty_2\rangle$ follows from the fact that $\langle x_1,x_2\rangle$ is non degenerate and that T is one-to-one continuous and with a range equal to E_1 . Let now

$$f(x) = \int_{E_1} e^{i\langle x,\alpha\rangle}\, d\mu(\alpha) \ ,$$

then

$$f(Ty) = \int_{E} e^{i\langle Ty,\alpha\rangle}\, d\mu(\alpha)$$

$$= \int_{E_2} e^{i\langle Ty,T\beta\rangle}\, d\nu(\beta) \ ,$$

where ν is the measure on E_2 induced by μ and the continuous transformation T^{-1} from E_1 to E_2 . Hence $f(Ty)$ is in $\mathcal{F}(E_2,\langle T\cdot,T\cdot\rangle)$ and

$$\widetilde{\int_{E_2}} e^{\frac{i}{2}\langle Ty,Ty\rangle} f(Ty)dy = \int_{E_2} e^{-\frac{i}{2}\langle T\beta,T\beta\rangle} d\nu(\beta)$$

$$= \int_{E_1} e^{-\frac{i}{2}\langle\alpha,\alpha\rangle} d\mu(\alpha) .$$

This then proves the proposition. □

<u>Proposition 4.4</u>

Let E be a real separable Banach space with a bounded symmetric non-degenerate bilinear form $\langle x,y\rangle$. Let $E = E_1 \oplus E_2$ be a splitting of E into two closed subspaces E_1 and E_2 such that $\langle x_1,x_2\rangle = 0$ for $x_1 \in E_1$ and $x_2 \in E_2$. Assume now that the restriction of the form $\langle x,y\rangle$ to $E_1 \times E_1$ is non degenerate, then the restriction to $E_2 \times E_2$ is also non degenerate and for any $f \in \mathcal{F}(E,\langle\,,\rangle)$ we have

$$\widetilde{\int_{E}} e^{\frac{i}{2}\langle x,x\rangle} f(x)dx = \widetilde{\int_{E_1}} e^{\frac{i}{2}\langle x_1,x_1\rangle} [\widetilde{\int_{E_2}} e^{\frac{i}{2}\langle x_2,x_2\rangle} f(x_1,x_2)dx_2]dx_1 ,$$

with $f(x_1,x_2) = f(x_1 \oplus x_2)$ for $x_1 \in E_1$ and $x_2 \in E_2$.

<u>Proof:</u> Let $x_2 \in E_2$ and assume that $\langle x_2,y_2\rangle = 0$ for all $y_2 \in E_2$. Since $E = E_1 \oplus E_2$ we then have that $\langle x_2,y\rangle = 0$ for all $y \in E$, hence $x_2 = 0$, and the form restricted to $E_2 \times E_2$ is non degenerate. Since now $E = E_1 \oplus E_2$, E is equivalent as a metric group with $E_1 \times E_2$ and therefore

$$f(x) = \int_{E_1 \times E_2} e^{i\langle x,\alpha_1+\alpha_2\rangle} d\mu(\alpha_1,\alpha_2) ,$$

hence

$$f(x_1,x_2) = \int e^{i\langle x_1,\alpha_1\rangle} \cdot e^{i\langle x_2,\alpha_2\rangle} d\mu(\alpha_1,\alpha_2) .$$

So that, for fixed x_1, $f(x_1,x_2) \in \mathcal{F}(E_2,\langle\,,\rangle)$ and

$$\widetilde{\int_{E_2}} e^{\frac{i}{2}\langle x_2,x_2\rangle} f(x_1,x_2)dx_2 = \int_{E_2\times E_1} e^{-\frac{i}{2}\langle \alpha_2,\alpha_2\rangle} e^{i\langle x_1,\alpha_1\rangle} d\mu(\alpha_1,\alpha_2).$$

We see that (4.39) is in $\mathcal{F}(E_1,\langle\ \rangle)$ and we compute

$$\widetilde{\int_{E_1}} e^{\frac{i}{2}\langle x_1,x_1\rangle} [\widetilde{\int_{E_2}} e^{\frac{i}{2}\langle x_2,x_2\rangle} f(x_1,x_2)dx_2]dx_1$$

$$= \int_{E_1\times E_2} e^{-\frac{i}{2}\langle \alpha_1,\alpha_1\rangle} e^{-\frac{i}{2}\langle \alpha_2,\alpha_2\rangle} d\mu(\alpha_1,\alpha_2)$$

$$= \int_E e^{-\frac{i}{2}\langle \alpha,\alpha\rangle} d\mu(\alpha) ,$$

and this proves the proposition. $\square$

In the case where E is a Hilbert space and one has $\langle x,y\rangle = (x,By)$, with B and B^{-1} both bounded, a stronger version of proposition 4.4 holds, we have namely:

Proposition 4.5.

Let $\mathcal{H}$ be a real separable Hilbert space and let $\langle x,y\rangle = (x,By)$, with B symmetric, such that $D(B) = R(B) = \mathcal{H}$, with a bounded inverse B^{-1}, then $\mathcal{F}(\mathcal{H},\langle\ \rangle) = \mathcal{F}(\mathcal{H})$. If $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}$ such that the restriction of $\langle x,y\rangle$ to $\mathcal{H}_1\times\mathcal{H}_1$ is non degenerate, and $\mathcal{H}_2 = B^{-1}\mathcal{H}_1^{\perp}$, where $\mathcal{H}_1^{\perp}$ is the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is a splitting of $\mathcal{H}$ in two closed subspaces such that the restriction of $\langle x,y\rangle$ to $\mathcal{H}_1\times\mathcal{H}_2$ is identically zero. Hence by the previous proposition the restriction of $\langle x,y\rangle$ to $\mathcal{H}_2\times\mathcal{H}_2$ is non degenerate and for any $f \in \mathcal{F}(\mathcal{H})$ we have

$$\tilde{\int_{\mathcal{H}_1}} e^{\frac{i}{2}\langle x,x\rangle} f(x)dx = \int_{\mathcal{H}_1} e^{\frac{i}{2}\langle x_1,x_1\rangle} [\tilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2}\langle x_2,x_2\rangle} f(x_1,x_2)dx_2]dx_1 ,$$

with $f(x_1,x_2) = f(x_1 \oplus x_2)$, where $x = x_1 \oplus x_2$ is the splitting of $\mathcal{H}$ into $\mathcal{H}_1 \oplus \mathcal{H}_2$ and the sum is orthogonal with respect to $\langle x,y\rangle = (x,By)$.

Proof. That $\mathcal{F}(\mathcal{H},\langle\ \rangle) = \mathcal{F}(\mathcal{H})$ was already proved by the formulas (4.34) and (4.35). So let now $\mathcal{H}_1$ and $\mathcal{H}_2$ be as in the proposition. Since B^{-1} has a bounded inverse B , $\mathcal{H}_2 = B^{-1}\mathcal{H}_1^{\perp}$ is obviously a closed subspace and $\langle x,y\rangle = 0$ for $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. Let $x \in \mathcal{H}$ be such that $\langle x,y\rangle = 0$ for all $y \in \mathcal{H}_1 + \mathcal{H}_2$. Then $\langle x,y\rangle = 0$ for all $y \in \mathcal{H}_1$, hence $x \in \mathcal{H}_2$. Now let $y \in \mathcal{H}_2$, then $y = B^{-1}z$ with $z \in \mathcal{H}_1^{\perp}$ and $0 = \langle x,y\rangle = (x,z)$ for all $z \in \mathcal{H}_1^{\perp}$, so that $x \in \mathcal{H}_1$, but since $x \in \mathcal{H}_2$ we have also $\langle x,y\rangle = 0$ for all y in $\mathcal{H}_1$. Therefore, by the non degeneracy of $\langle x,y\rangle$ in $\mathcal{H}_1 \times \mathcal{H}_1$, we have that $x = 0$. This shows that $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and that the orthogonal complement of $B(\mathcal{H}_1 + \mathcal{H}_2)$ is zero. Hence $B(\mathcal{H}_1 + \mathcal{H}_2)$ is dense and since B and B^{-1} both are bounded and therefore preserve the topology, we have that $\mathcal{H}_1 + \mathcal{H}_2$ is dense in $\mathcal{H}$. But since now both $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed and $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ we get that $\mathcal{H}_1 + \mathcal{H}_2 = \mathcal{H}$. This proves the proposition. $\square$

5. Feynman path integrals for the anharmonic oscillator.

By the anharmonic oscillator with n degrees of freedom we shall understand the mechanical system in R^n with classical action integral of the form:

$$S_t = \frac{m}{2}\int_0^t \dot{\gamma}(\tau)^2 d\tau - \frac{1}{2}\int_0^t \gamma A^2 \gamma d\tau - \int_0^t V(\gamma(\tau))d\tau \ , \tag{5.1}$$

where A^2 is a strictly positive definite matrix in R^n and $\dot{\gamma}(\tau) = \frac{d\gamma}{d\tau}$ and $V(x)$ is a nice function which in the following shall be taken to be in the space $\mathcal{F}(R^n)$ i.e.

$$V(x) = \int_{R^n} e^{i\alpha x} d\mu(\alpha) \ , \tag{5.2}$$

where μ is a bounded complex measure. We shall of course also assume, for physical reasons, that V is real.[1] Let $\varphi(x) \in \mathcal{F}(R^n)$ with

$$\varphi(x) = \int_{R^n} e^{i\alpha x} d\nu(\alpha) \ , \tag{5.3}$$

then we shall give a meaning to the Feynman path integral

$$\tilde{\int_{\gamma(t)=x}} e^{\frac{i}{2}(\int_0^t \dot{\gamma}^2 d\tau - \int_0^t \gamma A^2 \gamma d\tau)} \cdot e^{-i\int_0^t V(\gamma(\tau))d\tau} \varphi(\gamma(0))d\gamma \ , \tag{5.4}$$

by using the integral defined in the previous section. For simplicity we shall assume in what follows that $m = \hbar = 1$. In the previous section we only defined integrals over linear spaces, so we shall first have to transform the non homogeneous boundary condition $\gamma(t) = x$ into a homogeneous one. This is easily done if there exists a solution $\beta(\tau)$ to the following boundary value problem on the interval $[0,t]$:

$$\ddot\beta + A^2\beta = 0 \ , \quad \beta(t) = x \ , \quad \dot\beta(0) = 0 \ . \tag{5.5}$$

Let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of A . If we now assume that

$$t \neq (k+\tfrac{1}{2})\frac{\pi}{\lambda_i} \tag{5.6}$$

for all $k = 0,1,\dots,$ and $i = 1,\dots,n$, then (5.5) has a unique solution given by

$$\beta(\tau) = \frac{\cos A\tau}{\cos At}\, x \ . \tag{5.7}$$

We then make formally the substitution $\gamma \to \gamma + \beta$ in (5.4) and get

$$\widetilde{\int\limits_{\gamma(t)=0}} e^{\frac{i}{2}\int_0^t(\dot\gamma+\dot\beta)^2 d\tau - \frac{i}{2}\int_0^t(\gamma+\beta)A^2(\gamma+\beta)d\tau}\, e^{-i\int_0^t V(\gamma(\tau)+\beta(\tau))d\tau}\, \varphi(\gamma(0)+\beta(0))\,d\gamma \ . \tag{5.8}$$

Now, due to (5.5), we have that, if $\gamma(t) = 0$,

$$\begin{aligned}\int_0^t (\dot\gamma+\dot\beta)^2 d\tau - \int_0^t (\gamma+\beta)A^2(\gamma+\beta)d\tau &= \\ = \int_0^t \dot\gamma^2 d\tau - \int_0^t \gamma A^2\gamma d\tau + \beta(t)\dot\beta(t) \ .&\end{aligned} \tag{5.9}$$

Since $\beta(t) = x$ and $\dot\beta(t) = -A \operatorname{tg} At\, x$, we may write (5.8) as

$$e^{-\frac{i}{2}xA\operatorname{tg}Ax}\widetilde{\int\limits_{\gamma(t)=0}} e^{\frac{i}{2}\int_0^t(\dot\gamma^2-\gamma A^2\gamma)d\tau}\, e^{-i\int_0^t V(\gamma(\tau)+\beta(\tau))d\tau}\, \varphi(\gamma(0)+\beta(0))\,d\gamma \ . \tag{5.10}$$

Hence we have transformed the boundary condition to a homogeneous one. Let now $\mathcal{H}_0$ be the real separable Hilbert space of continuous functions γ from $[0,t]$ to R^n such that $\gamma(0) = 0$ and $|\gamma|^2 = \int_0^t \dot\gamma^2 d\tau$ is finite. In $\mathcal{H}_0$ the quadratic form $\int_0^t(\dot\gamma^2 - \gamma A^2\gamma)d\tau$ is obviously bounded and therefore given by a bounded symmetric operator B in $\mathcal{H}_0$, so that, with $(\gamma,\gamma) = |\gamma|^2$, we have

$$(\gamma,B\gamma) = \int_0^t (\dot{\gamma}^2 - \gamma A^2\gamma)d\tau \quad . \tag{5.11}$$

From the Sturm-Liouville theory we also have that B is onto with a bounded inverse B^{-1} , if $\lambda = 0$ is not an eigenvalue of the following eigenvalue problem on $[0,t]$:

$$\ddot{u} + A^2u = \lambda u, \; u(t) = 0, \; \dot{u}(0) = 0 \; . \tag{5.12}$$

One easely verifies that if t satisfies (5.6), then zero is not an eigenvalue for (5.12) and the Green's function for the eigenvalue problem (5.12) is given by

$$g_0(\sigma,\tau) = \frac{\cos A\sigma \sin A(t-\tau)}{A \cos At} \quad \text{for} \quad \sigma \leq \tau \; . \tag{5.13}$$

We shall now assume that t satisfies (5.6) i.e. that $\cos At$ is non degenerate. Then since the range of B is equal to all $\mathcal{H}_0$ and B^{-1} is bounded we are in the case where the space D is equal to $\mathcal{H}_0$ and Δ is uniquely given by $(\gamma,B^{-1}\gamma)$. Using then the notation introduced in (4.33) with $\langle\gamma_1,\gamma_2\rangle = (\gamma_1,B\gamma_2)$ i.e.

$$\langle\gamma_1,\gamma_2\rangle = \int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma)d\tau \; , \tag{5.14}$$

we verify exactly as in section 3 that

$$f(\gamma) = e^{-i\int_0^t V(\gamma(\tau)+\beta(\tau))d\tau} \varphi(\gamma(0)+\beta(0))$$

is in $\mathcal{F}(\mathcal{H}_0)$. Hence

$$e^{-\frac{i}{2}xA\,\mathrm{tg}\,tAx} \int_{\mathcal{H}_0} e^{\frac{i}{2}\langle\gamma,\gamma\rangle} f(\gamma)d\gamma \tag{5.15}$$

is well defined and we take (5.15) to be the definition of the Feynman path integral (5.4), which we shall now compute. As in section 3 we have that

$$f(\gamma) = \sum_{n=0}^{\infty} (-i)^n \int_{0\le t_1\le\dots\le t_n\le t}\cdots\int e^{i\sum_{j=0}^{n}\alpha_j\beta_j(t_j)}\, e^{i\sum_{j=0}^{n}\alpha_j\gamma(t_j)}\, d\nu(\alpha_0)\prod_{j=1}^{n} d\mu(\alpha_j)\,dt_j \tag{5.16}$$

where $t_0 = 0$, the sum converges strongly in $\mathcal{F}(\mathcal{H}_0)$ and the integrands are continuous in α_j and t_j in the weak $\mathcal{F}(\mathcal{H}_0)$ topology, hence weakly and therefore strongly integrable in $\mathcal{F}(\mathcal{H}_0)$. Since $\mathcal{F}_\Delta$ is continuous in $\mathcal{F}(D^*) = \mathcal{F}(\mathcal{H}_0)$ we may therefore commute the sum and the integrals in (5.16) with the integral in (5.15). Hence it suffices to compute

$$\tilde{\int_{\mathcal{H}_0}} e^{\frac{i}{2}\langle\gamma,\gamma\rangle}\, e^{i\sum\alpha_j\gamma(t_j)}\, d\gamma \,. \tag{5.17}$$

Now $\gamma_i(\sigma) \in \mathcal{H}_0^* = \mathcal{H}_0$, hence it is of the form $\gamma_i(\sigma) = \langle\gamma,\gamma_\sigma^i\rangle$ and therefore, by the definition, (5.17) is equal to

$$e^{-\frac{i}{2}\sum_{ij}\alpha_i\langle\gamma_{t_i},\gamma_{t_j}\rangle\alpha_j} \,. \tag{5.18}$$

Using now that (5.13) is the Green's function for (5.12) one verifies by standard computations that

$$\langle\gamma_\sigma,\gamma_\tau\rangle = g_0(\sigma,\tau) \,. \tag{5.19}$$

So we have computed (5.15) and shown that it is equal to

$$e^{-\frac{i}{2}xA\,\mathrm{tg}\,tA\,x}\sum_{n=0}^{\infty}(-i)^n \underset{0\leq t_1\leq\dots\leq t_n\leq t}{\int\dots\int} e^{\sum_{j=0}^{n}\alpha_j\beta(t_j)}$$

$$e^{-\frac{i}{2}\sum_{jk=0}^{n}\alpha_j g_o(t_j,t_k)\alpha_k}\,d\nu(\alpha_o)\prod_{j=1}^{n}d\mu(\alpha_j)dt_j \tag{5.20}$$

$$= e^{-\frac{i}{2}xA\,\mathrm{tg}\,tA\,x}\sum_{n=0}^{\infty}(-i)^n \underset{0\leq t_1\dots\leq t}{\int\dots\int} e^{i\sum_{j=0}^{n}\alpha_j\frac{\cos At_j}{\cos At}x}$$

$$e^{-\frac{i}{2}\sum_{jk=0}^{n}\alpha_j g_o(t_j,t_k)\alpha_k}\,d\nu(\alpha_o)\prod_{j=1}^{n}d\mu(\alpha_j)dt_j\ .$$

Let us now define $\Omega_o(x)$ by

$$\Omega_o(x) = |\tfrac{1}{\pi}A|^{1/4}\,e^{-\frac{1}{2}xAx}\ , \tag{5.21}$$

where $|\frac{1}{\pi}A|$ is the determinant of the transformation $\frac{1}{\pi}A$. It is well known that Ω_o is the normalized eigenfunction for the lowest eigenvalue of the self adjoint operator

$$H_o = -\tfrac{1}{2}\Delta + \tfrac{1}{2}\,xA^2x \tag{5.22}$$

in $L_2(R^n)$. We have in fact

$$H_o\Omega_o = \tfrac{1}{2}\,\mathrm{tr}\,A\,\Omega_o\ , \tag{5.23}$$

where $\mathrm{tr}\,A$ is the trace of A.

For $0\leq t_1\leq\dots\leq t_n\leq t$ we shall now compute the Feynman path integral

$$I(x) = \underset{\gamma(t)=x}{\tilde{\int}} e^{\frac{i}{2}\left(\int_0^t \dot{\gamma}^2(\tau)d\tau - \int_0^t \gamma A^2 \gamma d\tau\right)} e^{i\sum_{j=1}^n \alpha_j \gamma(t_j)} \Omega_0(\gamma(0))d\gamma$$

$$\underset{=}{\mathrm{Def}} \tilde{\int_{\mathcal{H}_0}} e^{\frac{i}{2}\int_0^t (\dot{\gamma}+\dot{\beta})^2 d\tau - \frac{i}{2}\int_0^t (\gamma+\beta)A^2(\gamma+\beta)d\tau} e^{i\sum_{j=1}^n \alpha_j(\gamma(t_j)+\beta(t_j))} \Omega_0(\gamma(0)+\beta(0))d\gamma \,. \tag{5.24}$$

By the previous calculation we have

$$I(x) = e^{-\frac{i}{2}xA\,\mathrm{tg}\,tAx} \int_{R^n} e^{i\sum_{j=0}^n \alpha_j \frac{\cos A t_j}{\cos A t}x} e^{-\frac{i}{2}\sum_{jk=0}^n \alpha_j g_0(t_j,t_k)\alpha_k} d\nu_0(\alpha_0), \tag{5.25}$$

where $t_0 = 0$ and $d\nu_0(\alpha_0)$ is given by

$$\Omega_0(x) = \int_{R^n} e^{ix\alpha_0} d\nu_0(\alpha_0) \,, \tag{5.26}$$

from which we get

$$d\nu_0(\alpha_0) = |4\pi^3 A|^{-1/4} e^{-\frac{1}{2}\alpha_0 A^{-1}\alpha_0} d\alpha_0 \,. \tag{5.27}$$

Substituting (5.27) in (5.25) we obtain

$$I(x) = e^{-\frac{i}{2}xA\,\mathrm{tg}\,tAx} e^{i\sum_{j=1}^n \alpha_j \frac{\cos A t_j}{\cos A t}x} e^{-\frac{i}{2}\sum_{jk=1}^n \alpha_j g_0(t_j,t_k)\alpha_k} \tag{5.28}$$

$$|4\pi^3 A|^{-1/4} \int_{R^n} e^{i\alpha_0 \frac{1}{\cos At}x} e^{-i\alpha_0 \sum_{j=1} g_0(0,t_j)\alpha_j} e^{-\frac{i}{2}\alpha_0 g(0,0)\alpha_0} e^{-\frac{1}{2}\alpha_0 A^{-1}\alpha_0} d\alpha_0 .$$

Now the integral above is equal to

$$|2\pi A \cos tA\, e^{-itA}|^{\frac{1}{2}} e^{-\frac{1}{2}xA(1-i\,\mathrm{tg}\,tA)x} e^{x\sum_{j=1}^n \frac{\sin A(t-t_j)e^{-itA}}{\cos At}\alpha_j} e^{-\frac{1}{2}\sum_{jk=1}^n \alpha_j \frac{\sin A(t-t_j)\sin A(t-t_k)}{A\cos At\, e^{itA}}\alpha_k} \,. \tag{5.29}$$

Hence we have finally that

$$I(x) = \left|\frac{1}{\sqrt{\pi}} A^{\frac{1}{2}} \cos tA\right|^{\frac{1}{2}} e^{-\frac{it}{2}\operatorname{tr} A} e^{-\frac{1}{2} xAx} e^{ix \sum_{j=1}^{n} e^{-i(t-t_j)A}\alpha_j} \tag{5.30}$$

$$\cdot\, e^{-\frac{i}{2} \sum_{jk=1}^{n} \alpha_j A^{-1} e^{-i(t-t_j)A} \sin A(t-t_k)\alpha_k} .$$

Now by an explicit calculation we obtain from (5.30) that

$$\int I(x)\Omega_o(x)dx = |\cos tA|^{\frac{1}{2}} e^{-\frac{it}{2}\operatorname{tr} A} e^{-\frac{1}{2} \sum_{jk=1}^{n} \alpha_j (2A)^{-1} e^{-i|t_k-t_j|A}\alpha_k} . \tag{5.31}$$

On the other hand it is well known from the standard theory o of the quantum mechanics for the harmonic oscillator that, for $0 \le t_1 \le \ldots \le t_n \le t$,

$$(\Omega_o, e^{i\alpha_1 x(t_1)} \ldots e^{i\alpha_n x(t_n)} e^{-itH_o} \Omega_o) =$$

$$= e^{-\frac{it}{2}\operatorname{tr} A} e^{-\frac{1}{2} \sum_{j,k=1}^{n} \alpha_j (2A)^{-1} e^{-i|t_j-t_k|A}\alpha_k} , \tag{5.32}$$

where $e^{i\alpha x(\tau)} = e^{i\tau H_o} e^{i\alpha x} e^{i\tau H_o}$. Hence we have proved the formula

$$|\cos At|^{-\frac{1}{2}} \int_{R^n} \Omega_o(x) \Big[\int_{\gamma(t)=x} e^{\frac{i}{2}\int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma) d\tau} e^{i \sum_{j=1}^{n} \alpha_j \gamma(t_j)} \Omega_o(\gamma(0)) d\gamma \Big] dx \tag{5.33}$$

$$= (\Omega_o, e^{i\alpha_1 x(t_1)} \ldots e^{i\alpha_n x(t_n)} e^{-itH_o} \Omega_o)$$

for $0 \le t_1 \le \ldots \le t_n \le t$.

Let now

$$H = H_o + V(x) . \tag{5.34}$$

We have the norm convergent expansion

$$e^{-itH} = \sum_{n=0}^{\infty} (-i)^n \int\limits_{0\le t_1 \le \dots \le t_n \le t} \!\!\cdots\!\! \int V(t_1)\dots V(t_n)\, e^{-itH_o}\, dt_1 \dots dt_n , \quad (5.35)$$

where $V(\tau) = e^{-i\tau H_o} V e^{i\tau H_o}$.

If now f,g and V are taken to be in $\mathcal{F}(R^n)$, then we get from (5.35), (5.34) and the fact that the sum and the integrals in (5.16) can be taken in the strong sense in $\mathcal{F}(\mathcal{H}_o)$, that

$$|\cos At|^{-\frac{1}{2}} \int_{R^n} f(x)\Omega_o(x)dx \Big[\int_{\gamma(t)=x} e^{\frac{i}{2}\int_0^t (\dot\gamma^2 - \gamma A^2 \gamma)d\tau}\, e^{-i\int_0^t V(\gamma(\tau))d\tau}\, g(\gamma(0))\Omega_o(\gamma(0))d\gamma\Big]dx \quad (5.36)$$

$$= (\Omega_o, f\, e^{-itH} g\Omega_o) .$$

This then, by the density of $\mathcal{F}(R^n)\Omega_o$ in $L_2(R^n)$, proves the formula

$$\psi(x,t) = |\cos At|^{-\frac{1}{2}} \widetilde{\int_{\gamma(t)=x}} e^{\frac{i}{2}\int_0^t (\dot\gamma^2 - \gamma A^2 \gamma)d\tau}\, e^{-i\int_0^t V(\gamma(\tau))d\tau}\, \varphi(\gamma(0))d\gamma \quad (5.37)$$

for the solution of the Schrödinger equation for the anharmonic oscillator

$$i \frac{\partial}{\partial t} \psi = -\tfrac{1}{2} \Delta\psi + \tfrac{1}{2} x A^2 x \psi + V(x)\psi \quad (5.38)$$

for all t such that $\cos At$ is non singular and $V \in \mathcal{F}(R^n)$ and the initial condition $\psi(x,0) = \varphi(x)$ is in $\mathcal{F}(R^n) \cap L_2(R^n)$. From (5.36) we only get (5.37) for $\varphi \in \mathcal{F}(R^n)\cdot\Omega_o$, but since the left hand side of (5.37) is continuous in L_2 as a function of φ and the right hand side for fixed x is continuous as a function of φ in $\mathcal{F}(R^n)$, by the fact that the sum and the integrals in (5.16) can be taken in the strong $\mathcal{F}(\mathcal{H}_0)$ sense, we get (5.37) for all $\varphi \in \mathcal{F}(R^n) \cap L_2(R^n)$. Although the integral over γ in (5.37) was defined by (5.15) using the translation by β, we have, since $D(B) = \mathcal{H}_0$, by proposition 4.2, that the integral in (5.15) is invariant under translations by any $\gamma_0 \in \mathcal{H}_0$ i.e. by any path $\gamma_0(t)$ for which $\gamma_0(t) = 0$ and the kinetic energy $\frac{1}{2}\int_0^t \dot{\gamma}_0(\tau)^2 d\tau$ is finite. Hence as a matter of fact the definition of the integral over γ in (5.37) does not depend on the specific choice of β. We state these results, for the case when m and $\hbar$ are not necessarily both equal to one, in the following theorem.

Theorem 5.1

Let $\mathcal{H}_0$ be the real separable Hilbert space of continuous functions γ from $[0,t]$ into R^n such that $\gamma(t) = 0$ and with finite kinetic energy $\frac{m}{2}\int_0^t \dot{\gamma}(\tau)^2 d\tau$, and norm given by $|\gamma|^2 = \int_0^t \dot{\gamma}(\tau)^2 d\tau$. Let B be the bounded symmetric operator on $\mathcal{H}$, with $D(B) = \mathcal{H}_0$, given by $(\gamma_1, B\gamma_2) = \langle \gamma_1, \gamma_2 \rangle$ with

$$\langle \gamma, \gamma \rangle = \frac{1}{\hbar}\int_0^t (m\dot{\gamma}(\tau)^2 - \gamma A^2 \gamma)\, d\tau ,$$

where A^2 is a strictly positive definite matrix in R^n. Then for all values of t such that $\cos At$ is a non singular transformation in R^n we have that the range of B is $\mathcal{H}_0$ and B^{-1} is a bounded symmetric operator on $\mathcal{H}_0$. Hence, by proposition 4.2, $D = \mathcal{H}_0$ and $\Delta(x,y)$ is uniquely given by B. Let $\beta(\tau)$ be any continuous path with $\beta(t) = x$ and finite kinetic energy, and let V and φ be in $\mathcal{F}(R^n)$, then

$$f(\gamma) = e^{\frac{i}{2}\langle\beta,\beta\rangle}\, e^{i\langle\gamma,\beta\rangle} \cdot e^{-i\int_0^t V(\gamma(\tau)+\beta(\tau))d\tau}\, \varphi(\gamma(0)+\beta(0))$$

is in $\mathcal{F}(\mathcal{H}_o)$ and

$$\widetilde{\int_{\mathcal{H}_0}} e^{\frac{i}{2}\langle\gamma,\gamma\rangle} f(\gamma)d\gamma \tag{5.39}$$

does not depend on β. Moreover if φ is also an $L_2(R^n)$ then

$$\psi(x,t) = |\cos A\,t|^{-\frac{1}{2}} \widetilde{\int_{\gamma(t)=x}} e^{\frac{i}{2\hbar}\int_0^t(m\dot{\gamma}^2-\gamma A^2\gamma)d\tau}\, e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau}\, \varphi(\gamma(0))d\gamma ,$$

where the integral over γ is defined by (5.39), and $\psi(x,t)$ is the solution of the Schrödinger equation for the anharmonic oscillator

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta\psi + \tfrac{1}{2} xA^2 x\psi + V(x)\psi$$

with initial values $\psi(x,0) = \varphi(x)$. $\square$

Let us now set, for $0 \le t_1 \le \ldots \le t_n \le t$, $\varphi(x) \in \mathcal{S}(R^n)$ and t such that $\cos At$ is non singular:

$$I(x) = \widetilde{\int_{\gamma(t)=x}} e^{\frac{i}{2}\int_0^t(\dot{\gamma}(\tau)^2-\gamma A^2\gamma)d\tau}\, e^{i\sum_{j=1}^n \alpha_j\gamma(t_j)}\, \varphi(\gamma(0))d\gamma \tag{5.40}$$

$$\underset{=}{\overset{\Delta}{\mathrm{Def}}} \int_{\mathcal{H}} e^{\frac{i}{2}\int_0^t((\dot{\gamma}+\dot{\beta})^2-(\gamma+\beta)A^2(\gamma+\beta))d\tau}\, e^{i\sum_{j=1}^n \alpha_j(\gamma(t_j)+\beta(t_j))}\, \varphi(\gamma(0)+\beta(0))d\gamma .$$

with Δ and $\mathcal{H}_o$ as given in the previous theorem, and $\beta(\tau)$ some path with finite kinetic energy such that $\beta(t) = x$. We then get in the same way as for (5.24) that

$$I(x) = e^{-\frac{i}{2}xA\,\mathrm{tg}\,tA\,x} \int_{R^n} e^{i\sum_{j=0}^n \alpha_j \frac{\cos At_j}{\cos At}x}\, e^{-\frac{i}{2}\sum_{jk=0}^n \alpha_j g_o(t_j,t_k)\alpha_k}\, \hat{\varphi}(\alpha_o)d\alpha_o \tag{5.41}$$

where $t_o = 0$, $g_o(\sigma,\tau)$ is given by (5.13) and

$$\varphi(x) = \int_{R^n} e^{ix\alpha_o} \hat{\varphi}(\alpha_o)\,d\alpha_o \,.$$

Since $\hat{\varphi} \in \mathcal{S}(R^n)$ we get from (5.41) that $I(x) \in \mathcal{S}(R^n)$, hence $I(x)$ is integrable and we have by direct computation, if $\sin At$ is non singular,

$$\int I(x)dx = \left|\frac{i}{2\pi} A \operatorname{tg} tA\right|^{-\frac{1}{2}} \int_{R^n} e^{\frac{i}{2}\sum_{jk=0}^{n} \alpha_j \frac{\cos At_j \cos At_k}{A \sin At \cos At} \alpha_k} e^{-\frac{i}{2}\sum_{jk=0}^{n} \alpha_j g(t_j,t_k)\alpha_k} \hat{\varphi}(\alpha_o)\,d\alpha_o \,.$$

Hence we have

$$\int I(x)dx = \left|\frac{i}{2\pi} A \operatorname{tg} tA\right|^{-\frac{1}{2}} \int_{R^n} e^{-\frac{i}{2}\sum_{jk=0}^{n} \alpha_j g(t_j,t_k)\alpha_k} \hat{\varphi}(\alpha_o)\,d\alpha_o \,, \quad (5.42)$$

where $g(\sigma,\tau) = g(\tau,\sigma)$ and

$$g(\sigma,\tau) = -\frac{\cos A\sigma \cos A(t-\tau)}{A \sin At} \,. \quad (5.43)$$

We now observe that $g(\sigma,\tau)$ is the Green's function for the self adjoint operator $-\frac{d^2}{d\tau^2} - A^2$ on $L_2([0,t];R^n)$ with Neumann boundary conditions $\dot{\gamma}(0) = \dot{\gamma}(t) = 0$.

Let now $\mathcal{H}$ be the real separable Hilbert space of continuous functions γ from $[0,t]$ to R^n with finite kinetic energy without any conditions on the boundary and with norm given by

$$|\gamma|^2 = \int_0^t (\dot{\gamma}^2 + \gamma^2)\,d\tau \,. \quad (5.44)$$

Let B_N be the bounded symmetric operator with $D(B_N) = \mathcal{H}$, given by $(\gamma_1, B_N\gamma_2) = \langle\gamma_1,\gamma_2\rangle$ with

$$\langle\gamma,\gamma\rangle = \int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma)\,d\tau \,. \quad (5.45)$$

Then if $\sin At$ is non singular we have from (5.43) that $R(B_N) = \mathcal{H}$ and B_N^{-1} is bounded. Hence with $D = B_N$ the form Δ_N of the previous section is uniquely given by B_N and $\Delta_N(\gamma_1,\gamma_2) = (\gamma_1, B_N^{-1}\gamma_2)_N$. Define now $\gamma_\sigma \in \mathcal{H} \times R^n$ by

$$\langle \gamma, \gamma_\sigma \rangle = \gamma(\sigma) , \tag{5.46}$$

then we get, by using the fact that $g(\sigma,\tau)$ is the Green's function for the Neumann boundary conditions, that

$$\langle \gamma_\sigma, \gamma_\tau \rangle = g(\sigma,\tau) . \tag{5.47}$$

From this we obtain, for $0 \leq t_1 \leq \ldots \leq t_n \leq t$

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\langle\gamma,\gamma\rangle} e^{i\Sigma\alpha_j\gamma(t_j)} d\gamma = e^{-\frac{i}{2}\sum\limits_{j,k=1}^{n}\alpha_j g(t_j,t_k)\alpha_k} . \tag{5.48}$$

Let us now define for any $f(\gamma) \in \mathcal{F}(\mathcal{H})$ the path integral

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\int_0^t (\dot{\gamma}^2 - \gamma A^2\gamma)d\tau} f(\gamma)d\gamma \overset{\text{Def}}{=} \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\langle\gamma,\gamma\rangle} f(\gamma)d\gamma . \tag{5.49}$$

We then have that

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\int_0^t (\dot{\gamma}^2 - \gamma A^2\gamma)d\tau} e^{i\Sigma\alpha_j\gamma(t_j)} d\gamma = e^{-\frac{i}{2}\sum\limits_{j,k=1}^{n}\alpha_j g(t_j,t_k)\alpha_k} . \tag{5.50}$$

Let now t be such that $\sin At$ and $\cos At$ both are non singular, and let us assume that $f(\gamma)$ is in $\mathcal{F}(\mathcal{H})$. Then one easely verifies that

$$e^{i\int_0^t (\dot{\gamma}\dot{\beta} - \gamma A^2\beta)d\tau} f(\gamma+\beta) \tag{5.51}$$

is in $\mathcal{F}(\mathcal{H}_0)$ for any $\beta \in \mathcal{H}$, where $\mathcal{H}_0$ was defined as the space of paths γ with finite kinetic energy and such that $\gamma(t) = 0$. From (5.50) and (5.42) we now easely get the following formula, if we use the fact that the linear functionals $\gamma \to \gamma(\sigma)$ span a dense subset of $\mathcal{H}_0$ as well as of $\mathcal{H}$:

$$\left|\frac{i}{2\pi}A \operatorname{tg} tA\right|^{\frac{1}{2}} \int_{R^n} dx \widetilde{\int_{\gamma(t)=x}} e^{\frac{i}{2}\int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma) d\tau} f(\gamma) d\gamma = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma) d\tau} f(\gamma) d\gamma . \tag{5.52}$$

By the same method as used in the proof of theorem 5.1 we now have that, if φ_1, φ_2 and V are in $\mathcal{F}(R^n)$, then

$$f(\gamma) = \overline{\varphi_1}(\gamma(t))\, e^{-i\int_0^t V(\gamma(\tau)) d\tau} \varphi_2(\gamma(0)) \tag{5.53}$$

is in $\mathcal{F}(\mathcal{H})$, hence by (5.52) and theorem 5.1 we get

$$(\varphi_1, e^{-itH}\varphi_2) = \left|\frac{i}{2\pi}A \sin At\right|^{-\frac{1}{2}} \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}\int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma) d\tau}\, e^{-i\int_0^t V(\gamma(\tau)) d\tau} \overline{\varphi}_1(\gamma(t)) \varphi_2(\gamma(0)) d\gamma . \tag{5.54}$$

We have proved this formula only for values of t for which both $\cos At$ and $\sin At$ are non singular. However from (5.43) we see that, for fixed σ and τ, $g(\sigma,\tau)$ is a continuous function of t for values of t for which $\sin At$ is non singular. From this it easely follows that the right hand side of (5.54) is a continuous function of t for values of t for which $\sin At$ is non singular. Since the left hand side of (5.54) is a continuous function of t we get that (5.54) holds for all values of t for which $\sin At$ is non singular. We summarize these results in the following theorem, for the case when $\hbar$ and m are not necessarily equal to 1:

Theorem 5.2

Let $\mathcal{H}$ be the separable real Hilbert space of continuous functions γ from $[0,t]$ to R^n, such that the kinetic energy is finite with norm given by $|\gamma|^2 = \int_0^t (\dot\gamma^2 + \gamma^2)d\tau$. Let B_N be the bounded symmetric operator on $\mathcal{H}$ with $D(B_N) = \mathcal{H}$ given by $(\gamma_1, B_N\gamma_2) = \langle\gamma_1,\gamma_2\rangle$, with

$$\langle\gamma,\gamma\rangle = \frac{1}{\hbar}\int_0^t (m\dot\gamma(\tau)^2 - \gamma A^2\gamma)d\tau \ ,$$

where A^2 is a strictly positive definite matrix in R^n. Then for all values of t such that $\sin At$ is non singular we have that the range of B_N is $\mathcal{H}$ and B_N^{-1} is a bounded symmetric operator on $\mathcal{H}$. Hence, by proposition 4.2, $D = \mathcal{H}$ and Δ_N is uniquely given by B_N. Let now φ_1, φ_2 and V be in $\mathcal{F}(R^n)$ then

$$f(\gamma) = e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau} \ \bar\varphi_1(\gamma(t))\varphi_2(\gamma(0))$$

is in $\mathcal{F}(\mathcal{H})$ and

$$(\varphi_1, e^{-itH}\varphi_2) = \left|\frac{i}{2\pi}A\sin At\right|^{-\frac{1}{2}} \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\int_0^t (m\dot\gamma^2 - \gamma A^2\gamma)d\tau} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau} \bar\varphi_1(\gamma(t))\varphi_2(\gamma(0))d\gamma \ .$$

Remark: If A^2 is not necessarily positive definite but only non degenerate as a transformation in R^n, then with A as the unique square rooth of A^2 with non negative imaginary part both Theorem 5.1 and 5.2 still hold in the following sense. The Feynman path integrals are still well defined for all values of t such that $\cos At$ respectively $\sin At$ are non singular, and we may use proposition (4.5) to prove that the operators so defined form a semigroup under t, and we can also prove, since the expansion in powers of V converges, that it is a group of unitary operators in $L_2(R^n)$, by using the same method as in [30],2). Further, by computing directly the derivative with respect to t of $(\varphi_1, e^{-itH}\varphi_2)$ as given in theorem 5.2, we get that the infinitesimal generator for this unitary group is actually a self adjoint extension of

$$(-\Delta + xA^2x + V(x))$$

defined on $\mathcal{S}(R^n)$. The point of this remark is to show that the theory of Fresnel integrable functions applied to quantum mechanics via Feynman path integral has much wider applications than the more classical treatment by analytic continuation from real to imaginary time and then treating the corresponding heat equation by integration over the Wiener measure space.

6. Expectations with respect to the ground state of the harmonic oscillator.

We consider a harmonic oscillator with a finite number of degrees of freedom. The classical action for the time interval $[0,t]$ is given by (5.1) with $V = 0$. The corresponding action for the whole trajectory is given by

$$S_0(\gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty} \dot{\gamma}(\tau)^2 d\tau - \tfrac{1}{2}\int_{-\infty}^{\infty} \gamma A^2 \gamma \, d\tau \ , \tag{6.1}$$

where $\gamma(\tau)$ and A^2 are as in (5.1) and we have set, for typographical reasons, $m = 1$. Let now $\mathcal{H}$ be the real Hilbert space of real square integrable functions on R with values in R^n and norm given by

$$|\gamma|^2 = \int_{-\infty}^{\infty} \dot{\gamma}(\tau)^2 d\tau + \int_{-\infty}^{\infty} \gamma(\tau)^2 d\tau \ . \tag{6.2}$$

Let B be the symmetric operator in $\mathcal{H}$ given by

$$(\gamma,B\gamma) = \int_{-\infty}^{\infty}(\dot{\gamma}(\tau)^2 - \gamma A^2 \gamma)d\tau \tag{6.3}$$

with domain $D(B)$ equal to the functions γ in $\mathcal{H}$ with compact support. We then have, for any $\gamma \in D(B)$, that

$$S_0(\gamma) = \tfrac{1}{2}(\gamma,B\gamma) \ , \tag{6.4}$$

where $(\,,)$ is the inner product in $\mathcal{H}$. The Fourier transform of an element γ in $\mathcal{H}$ is given by

$$\hat{\gamma}(p) = 1/\sqrt{2\pi}\int_{-\infty}^{\infty} e^{ipt}\, \gamma(t)dt \tag{6.5}$$

and the mapping $\gamma \to \hat{\gamma}$ is then an isometry of $\mathcal{H}$ onto the real subspace of functions in $L_2((p^2+1)dp)$ satisfying

$$\overline{\hat{\gamma}(p)} = \hat{\gamma}(-p) \tag{6.6}$$

and we have, for any $\gamma \in D(B)$,

$$S_0(\gamma) = \tfrac{1}{2}(\gamma, B\gamma) = \int_R \overline{\hat{\gamma}(p)}(\tfrac{1}{2}\, p^2 - \tfrac{1}{2}\, A^2)\hat{\gamma}(p)dp \,. \tag{6.7}$$

Moreover the range $R(B)$ of B consists of functions whose Fourier transformsare smooth functions and in $L_2[(p^2+1)dp]$. Let D be the real Banach space of functions in $\mathcal{H}$ whose Fourier transforms are continuously differentiable functions with norm given by

$$\|\gamma\| = |\gamma| + \sup_p \left|\frac{d\hat{\gamma}}{dp}(p)\right| \,. \tag{6.8}$$

We have obviously that the norm in D is stronger than the norm in $\mathcal{H}$ and that D contains the range of B . We now define on $D \times D$ the symmetric form

$$\Delta(\gamma_1,\gamma_2) = \lim_{\epsilon \searrow 0} \int_R \overline{\hat{\gamma}_1(p)}(p^2 - A^2 + i\epsilon)^{-1}\, \hat{\gamma}_2(p)(p^2+1)^2 dp. \tag{6.9}$$

That this limit exists follows from the fact that $\overline{\hat{\gamma}_1(p)}\hat{\gamma}_2(p)$ is continuously differentiable and in $L_1[(p^2+1)dp]$. **That** the form is continuous and bounded on $D \times D$ follows by standard results and (6.8). That the form is symmetric,

$$\Delta(\gamma_1,\gamma_2) = \Delta(\gamma_2,\gamma_1) \,,$$

follows from (6.9) and (6.6). In fact the limit (6.9) is given by

$$\begin{aligned}\Delta(\gamma_1,\gamma_2) = {} & P\int_R \overline{\hat{\gamma}_1(p)}(p^2 - A^2)^{-1}\, \hat{\gamma}_2(p)(p^2+1)^2 dp \\ & - i\pi \int \overline{\hat{\gamma}_1(p)}\delta(p^2 - A^2)\hat{\gamma}_2(p)(p^2+1)^2 dp \,,\end{aligned} \tag{6.10}$$

where the first integral is the principal value and hence real by (6.6). We see therefore that

$$\operatorname{Im} \Delta(\gamma,\gamma) \leq 0 \ . \tag{6.11}$$

Let now $\gamma_1 \in D$ and $\gamma_2 \in D(B)$, then

$$\Delta(\gamma_1,B\gamma_2) = \lim_{\varepsilon \searrow 0} \int_R \overline{\hat{\gamma}_1(p)}(p^2 - A^2 + i\varepsilon)^{-1}(p^2 - A^2)\hat{\gamma}_2(p)(p^2+1)dp$$

$$= \int \overline{\hat{\gamma}_1(p)}\hat{\gamma}_2(p)(p^2+1)dp \ .$$

So that

$$\Delta(\gamma_1,B\gamma_2) = (\gamma_1,\gamma_2) \ . \tag{6.12}$$

We have now verified that $\mathcal{H}$, D, B and Δ satisfy the conditions in the definition 4.0 for the Fresnel integral with respect to Δ .

Hence for any function $f \in \mathcal{F}(D^*)$ we have that

$$\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(\gamma,B\gamma)} f(\gamma)d\gamma \tag{6.13}$$

is well defined and given by (4.12). It follows from (6.8) that γ_t , given by

$$(\gamma_t,\gamma) = \gamma(t) \ ,$$

is in $D \times R^n$, since

$$\hat{\gamma}_t(p) = \sqrt{\frac{1}{2\pi}} \cdot \frac{e^{ipt}}{p^2+1} \ . \tag{6.14}$$

So that

$$f(\gamma) = e^{i\sum_{j=1}^{n} \alpha_j \cdot \gamma(t_j)} \tag{6.15}$$

is in $\mathcal{F}(D^*)$.

Hence we may compute (6.13) with $f(\gamma)$ given by (6.15) and we get

$$\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(\gamma,B\gamma)} f(\gamma)d\gamma = e^{-\frac{i}{2}\sum_{jk=1}^{n}\alpha_j \Delta(\gamma_{t_j},\gamma_{t_k})\alpha_k} . \tag{6.16}$$

From the definition of Δ we easely compute

$$\Delta(\gamma_s,\gamma_t) = \frac{1}{2iA} e^{-i|t-s|A} . \tag{6.17}$$

Hence we get that

$$\int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(\gamma,B\gamma)} e^{i\sum_{j=1}^{n}\alpha_j\gamma(t_j)} d\gamma = e^{-\frac{i}{2}\sum_{j,k=1}^{n}\alpha_j(2A)^{-1}e^{-i|t_j-t_k|A}\alpha_k} . \tag{6.18}$$

Let now Ω_o be the vacuum i.e. the function given by 5.21, and let us set in this section

$$H_o = -\tfrac{1}{2}\Delta + \tfrac{1}{2}xA^2x - \tfrac{1}{2}\operatorname{tr} A , \tag{6.19}$$

where we have changed the notation so that

$$H_o\Omega_o = 0 . \tag{6.20}$$

Let $t_1 \leq \ldots \leq t_n$, then we get from (6.18) and (5.32) that

$$\begin{aligned}(\Omega_o, e^{i\alpha_1 x(t_1)} \ldots e^{i\alpha_n x(t_n)}\Omega_o) &= \int_{\mathcal{H}}^{\Delta} e^{\frac{i}{2}(\gamma,B\gamma)} e^{i\sum_{j=1}^{n}\alpha_j\gamma(t_j)} d\gamma \\ &= \int_{\mathcal{H}}^{\Delta} e^{i\int_{-\infty}^{\infty}(\frac{1}{2}\dot{\gamma}^2-\frac{1}{2}\gamma A^2\gamma)d\tau} e^{i\sum_{j=1}^{n}\alpha_j\gamma(t_j)} d\gamma ,\end{aligned} \tag{6.21}$$

where

$$e^{i\alpha x(t)} = e^{-itH_o} e^{i\alpha x} e^{itH_o} .$$

<u>Theorem 6.1</u>

Let $\mathcal{H}$ be the real Hilbert space of real continuous and square integrable functions such that the norm given by

$$|\gamma|^2 = \int_{-\infty}^{\infty} \dot{\gamma}(\tau)^2 d\tau + \int_{-\infty}^{\infty} \gamma(\tau)^2 d\tau$$

is finite. Let B be the symmetric operator with domain equal to the functions in $\mathcal{H}$ with compact support and given by

$$(\gamma, B\gamma) = 2S_0(\gamma) = \int_{-\infty}^{\infty} (\dot{\gamma}(\tau)^2 - \gamma A^2 \gamma)\, d\tau \ ,$$

and let D be the real Banach space of functions in $\mathcal{H}$ with differentiable Fourier transforms and norm given by (6.8), and let Δ be given by (6.9). Then $(\mathcal{H}, D, B, \Delta)$ satisfies the condition of definition 4.0 for the integral normalized with respect to Δ. Let f, g and V be in $\mathcal{F}(R^n)$, then $f(\gamma(0))$ $g(\gamma(t))$ and $\exp[-i \int_0^t V(\gamma(\tau))d\tau]$ are all in $\mathcal{F}(D^*)$ and

$$(f\Omega_0,\ e^{-itH} g\Omega_0) = \int_{\mathcal{H}}^{\Delta} e^{iS_0(\gamma)}\ e^{-i\int_0^t V(\gamma(\tau))d\tau}\ \overline{f}(\gamma(0)) g(\gamma(t)) d\gamma \ ,$$

where

$$H = H_0 + V \ .$$

Proof: The first part of the theorem is already proved. Let therefore f be in $\mathcal{F}(R^n)$ i.e.

$$f(x) = \int e^{i\alpha x}\, d\nu(\alpha) \ , \tag{6.22}$$

then

$$f(\gamma(0)) = \int e^{i\alpha\gamma(0)}\, d\nu(\alpha) \ ,$$

which is in $\mathcal{F}(D^*)$ by the definition of $\mathcal{F}(D^*)$, since $\gamma(0) = (\gamma_0, \gamma)$ and we already proved that $\gamma_0 \in D$. Hence also $g(\gamma(t))$ is in $\mathcal{F}(D^*)$. Now

$$\int_0^t V(\gamma(\tau))d\tau = \int_0^t\!\!\int e^{i\alpha\gamma(\tau)}\, d\mu(\alpha)d\tau \tag{6.23}$$

is again in $\mathcal{F}(D^*)$ and therefore also $\exp[-i\int_0^t V(\gamma(\tau))d\tau]$

belongs to $\mathcal{F}(D^*)$ by proposition 4.1 (which states that $\mathcal{F}(D^*)$ is a Banach algebra). Since, also by proposition 4.1, the Fresnel integral with respect to Δ is a continuous linear functional on this Banach algebra we have

$$\begin{aligned}\int_{\mathcal{H}}^{\Delta} & e^{iS_0(\gamma)} e^{-i\int_0^t V(\gamma(\tau))d\tau} \bar{f}(\gamma(0))g(\gamma(t))d\gamma \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \ldots \int_0^t \int_{\mathcal{H}}^{\Delta} e^{iS_0(\gamma)} V(\gamma(t_1))\ldots V(\gamma(t_n))d\gamma\, dt_1\ldots dt_n\ .\end{aligned} \tag{6.24}$$

Utilizing now (6.23), (6.21) and the perturbation expansion (5.35) the theorem is proved. □

Theorem 6.2

Let the notations be the same as in theorem 6.1, and let $t_1 \leq \ldots \leq t_m$, then for $f_i \in \mathcal{S}(R^n)$, $i = 1,\ldots m$

$$(\Omega_o, f_1\, e^{-i(t_2-t_1)H} f_2\, e^{-i(t_3-t_2)H} f_3 \ldots e^{-i(t_m-t_{m-1})H} f_n\Omega_o)$$

$$= \int_{\mathcal{H}}^{\Delta} e^{iS_0(\gamma)} e^{-i\int_{t_1}^{t_m} V(\gamma(\tau))d\tau} \prod_{j=1}^{m} f_j(\gamma(t_j))d\gamma\ .$$

This theorem is proved by the series expansion of the function $\exp(-i\int_{t_1}^{t_m} V(\gamma(\tau))d\tau)$ and the fact that this series converges in $\mathcal{F}(D^*)$, in the same way as in the proof of theorem 6.1.

□

7. Expectations with respect to the Gibbs state of the harmonic oscillator.

Let $\mathcal{H}$, D, B and H_o be as in the previous section and define the continuous symmetric form on $D \times D$ by

$$\Delta_\beta(\gamma_1,\gamma_2) = P\int_R \overline{\hat{\gamma}_1(p)}(p^2-A^2)^{-1}\ \hat{\gamma}_2(p)(p^2+1)^2dp$$
$$- i\pi\int_R \overline{\hat{\gamma}_1(p)}\ \operatorname{cotgh}(\tfrac{\beta}{2}A)\cdot\delta(p^2-A^2)\hat{\gamma}_2(p)(p^2+1)^2dp\ , \tag{7.1}$$

where $\beta > 0$ and $\operatorname{cotgh}\frac{\beta}{2}A = (1-e^{-\beta A})^{-1}(1+e^{-\beta A})$. We see that $\Delta(\gamma_1,\gamma_2) = \Delta_\infty(\gamma_1,\gamma_2)$, and we verify in the same way as in the previous section that $\mathcal{H}$, D, B and Δ_β satisfy the conditions of definition 4.0 for the integral on $\mathcal{H}$ normalized with respect to Δ_β, so that we have in particular that

$$\operatorname{Im}\Delta_\beta(\gamma,\gamma) \leq 0 \tag{7.2}$$

and, for $\gamma_2 \in D(B)$,

$$\Delta_\beta(\gamma_1,B\gamma_2) = (\gamma_1,\gamma_2)\ . \tag{7.3}$$

From a direct computation we get, with γ_s defined as in the previous section (6.14), that

$$\Delta_\beta(\gamma_s,\gamma_t) = (2Ai(1-e^{-\beta A}))^{-1}[e^{-i|t-s|A} + e^{-\beta A}\ e^{i|t-s|A}]. \tag{7.4}$$

Set now

$$g_\beta(s-t) = \Delta_\beta(\gamma_s,\gamma_t)\ . \tag{7.5}$$

We may then compute the Fresnel integral

$$\int_{\mathcal{H}}^{\Delta_\beta} e^{iS_o(\gamma)}\ e^{i\sum_{j=1}^{m}\alpha_j\gamma(t_j)}\ d\gamma = e^{-\frac{i}{2}\sum_{j,k=1}^{m}\alpha_j g_\beta(t_j-t_k)\alpha_k}\ . \tag{7.6}$$

From theorem (2.1) of Ref. and the formula (2.24) of the same reference we then have, for $t_1 \leq \ldots \leq t_m$,

$$\int_{\mathcal{H}}^{\Delta_\beta} e^{iS_o(\gamma)} e^{i\sum_{j=1}^{m}\alpha_j\gamma(t_j)} d\gamma = (\mathrm{tr}\, e^{-\beta H_o})^{-1}\, \mathrm{tr}(e^{i\alpha_1 x(t_1)} \ldots e^{i\alpha_m x(t_m)} e^{-\beta H_o}) , \tag{7.7}$$

where $e^{i\alpha x(t)} = e^{-itH_o} e^{i\alpha x} e^{itH_o}$. And from this it follows that, for $f_i \in \mathcal{S}(R^n)$, $i = 1,\ldots,n$ and with $t_1 \leq \ldots \leq t_m$,

$$\omega_\beta^0(f_1(t_1)\ldots f_m(t_m)) = \int_{\mathcal{H}}^{\Delta_\beta} e^{iS_o(\gamma)} \prod_{j=1}^{m} f_j(\gamma(t_j))d\gamma , \tag{7.8}$$

where ω_β^0 is the Gibbs state of the harmonic oscillator i.e. for any bounded operator C on $L_2(R^n)$

$$\omega_\beta^0(C) = (\mathrm{tr}\, e^{-\beta H_o})^{-1}\, \mathrm{tr}(C\, e^{-\beta H_o}) . \tag{7.9}$$

Theorem 7.1

Let $t_1 \leq \ldots \leq t_m$ and $f_i \in \mathcal{F}(R^n)$, $i = 1,\ldots,m$. If

$$H = H_o + V$$

with $V \in \mathcal{F}(R^n)$ then

$$\omega_\beta^0(f_1\, e^{-i(t_2-t_1)H} f_2 \ldots e^{-i(t_m-t_{m-1})H} f_m) = \int_{\mathcal{H}}^{\Delta_\beta} e^{iS_o(\gamma)} e^{-i\int_{t_1}^{t_m} V(\gamma(\tau))d\tau} \prod_{j=1}^{m} f_j(\gamma(t_j))d\gamma .$$

Proof: This theorem is again proved by series expansion of the function $\exp(-i\int_{t_1}^{t_m} V(\gamma(\tau))d\tau)$ and the fact that this series converges in $\mathcal{F}(D^*)$ in complete analogy with the proof of theorem 6.1 and 6.2. $\square$

Let now $0 < f(\lambda) < 1$ be a positive continuous function defined on the positive real axis, and let us define, in conformity with (7.1), the symmetric continuous form on $D \times D$

$$\Delta_f(\gamma_1,\gamma_2) = P\int_R \overline{\hat{\gamma}_1(p)}(p^2-A^2)^{-1}\,\hat{\gamma}_2(p)(p^2+1)^2dp$$
$$- i\pi\int_R \hat{\gamma}_1(p)\,\frac{1+f(A)}{1-f(A)}\,\delta(p^2-A^2)\hat{\gamma}_2(p)(p^2+1)^2dp\,, \tag{7.10}$$

so that Δ_β is equal to Δ_f for $f(\lambda) = e^{-\beta\lambda}$.

It follows again easely that $\mathcal{H}$, D, B and Δ_f satisfy the conditions of definition 4.0 for the integral normalized with respect to Δ_f, and by computation we get

$$\Delta_f(\gamma_s,\gamma_t) = (2Ai(1-f(A)))^{-1}[e^{-i|t-s|A} + f(A)\;e^{i|t-s|A}]. \tag{7.11}$$

Therefore

$$\int_{\mathcal{H}}^{\Delta_f} e^{iS_o(\gamma)}\, e^{i\sum_{j=1}^{m}\alpha_j\gamma(t_j)}\, d\gamma = e^{-\frac{i}{2}\sum_{jk}\alpha_j g_f(t_j-t_k)\alpha_k}\,, \tag{7.12}$$

with $g_f(s-t) = \Delta_f(\gamma_s,\gamma_t)$.

Since $e^{i\alpha x(t)} = e^{-itH_o}\, e^{i\alpha x}\, e^{itH_o}$ span the called Weyl algebra on R^n as t and α varies, because $x(t) = \cos At\cdot x + i\,\frac{\sin At}{A}\pi$, where $\pi = \frac{1}{i}\frac{d}{dt}x(t)$ at $t = 0$, we may define a linear functional on the Weyl algebra by setting, for $t_1 \leq \dots \leq t_m$

$$\omega_f^0(e^{i\alpha_1 x(t_1)} \dots e^{i\alpha_m x(t_m)}) = e^{-\frac{1}{2}\Sigma\,\alpha_j g_f(t_i-t_j)\alpha_k}\,. \tag{7.13}$$

We can verify that (7.13) is consistent with the commutation relations for the Weyl algebra, and moreover ω_f^0 defines a normalized positive definite state on the Weyl algebra, which is a quasi free state in the terminology of Ref. [34], 2), 3).

In fact any quasifree state invariant under the group of automorphisms induced by e^{-itH_o} is of this form, and by (7.12) we have

$$\omega_f^0(e^{i\alpha_1 x(t_1)} \dots e^{i\alpha_m x(t_m)}) = \int_{\mathcal{H}}^{\Delta_f} e^{iS_o(\gamma)} e^{i\sum_{j=1}^{m}\alpha_j\gamma(t_j)} d\gamma . \tag{7.14}$$

We shall return to these considerations in greater details in the next section.

8. The invariant quasi-free states.

In this section we consider the harmonic oscillator with an infinite number of degrees of freedom. Hence let h be a real separable Hilbert space and A^2 be a positive self-adjoint operator on h such that zero is not an eigenvalue of A^2. The harmonic oscillator in h with harmonic potential $\frac{1}{2}x \cdot A^2 x$, where $x \cdot y$ is the inner product in h, has the classical action given by

$$S(\gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty} \dot{\gamma}(\tau)^2 d\tau - \tfrac{1}{2}\int_{-\infty}^{\infty} \gamma(\tau) \cdot A^2 \gamma(\tau) d\tau , \tag{8.1}$$

where $\dot{\gamma}(\tau)^2 = \dot{\gamma}(\tau) \cdot \dot{\gamma}(\tau)$, and $\dot{\gamma}(\tau)$ is the strong derivative in h of the trajectory $\gamma(\tau)$, where $\gamma(\tau)$ is a continuous and differentiable function from R to h. The corresponding quantum mechanical system is well known and easiest described in terms of the so called annihilation-creation operators.[1)] The Hamiltonian is formally given by

$$H_0 = -\tfrac{1}{2}\Delta + \tfrac{1}{2}x \cdot A^2 x - \tfrac{1}{2}\operatorname{tr} A , \tag{8.2}$$

where H_0 is so normalized that

$$H_0 \Omega_0 = 0 , \tag{8.3}$$

Ω_0 being the ground state of the harmonic oscillator. The precise definition of (8.2) is in terms of annihilation-creation operators as follows.

Let $\varphi(y)$ be the self adjoint operator which is the quantization of the function $y \cdot x$ on h and $\pi(y)$ its canonical conjugate, then $\varphi(x)$ and $\pi(x)$ is given in terms of the annihilation-creation operators a^* and a by

$$\varphi(x) = a^*((2A)^{-\frac{1}{2}}x) + a((2A)^{-\frac{1}{2}}x)$$
$$\pi(x) = i[a^*((\tfrac{1}{2}A)^{\frac{1}{2}}x) - a((\tfrac{1}{2}A)^{\frac{1}{2}}x)] \tag{8.4}$$

for x in the domain of $A^{-\frac{1}{2}}$ and $A^{\frac{1}{2}}$ respectively. The annihilation-creation operators $a^*(x)$ and $a(x)$ are linear in x and defined for all $x \in h$, and satisfy

$$[a(x),a(y)] = [a^*(x),a^*(y)] = 0 \quad \text{and}$$
$$[a(x),a^*(y)] = x\cdot y \,, \tag{8.5}$$

which together with (8.4) gives

$$[\pi(x),\pi(y)] = [\varphi(x),\varphi(y)] = 0 \quad \text{and}$$
$$[\pi(x),\varphi(y)] = \frac{1}{i}\, x\cdot y \,. \tag{8.6}$$

A representation of/algebra generated by the annihilation-creation operators is provided by introducing a cyclic element Ω_o such that

$$a(x)\Omega_o = 0 \tag{8.7}$$

for all $x \in h$.

Let h^c be the complexification of h. We extend by linearity a^* and a to h^c and define the self adjoint operator

$$B_o(z) = a^*(z) + a(\bar{z}) \,, \tag{8.8}$$

where $z = x + iy \in h^c$ and $\bar{z} = x - iy$ if x and y are in h. We then have the commutation relations

$$[B_o(z_1),B_o(z_2)] = i\sigma(z_1,z_2) \,, \tag{8.9}$$

where $\sigma(z_1,z_2) = \mathrm{Im}(z_1,z_2)$, and $(z_1,z_2) = x_1\cdot x_2 + y_1\cdot y_2$ is the inner product in h^c. (8.7) then gives us the so called free Fock representation of the canonical commutation relations.

The Weyl algebra over h^c is the *-algebra generated by elements $e(z)$ with $z \in h^c$, where the *-operator is given by $e(z)^* = e(-z)$ and the multiplication is given by

$$e(z_1)\cdot e(z_2) = e^{-i\sigma(z_1,z_2)}\, e(z_1+z_2) \ , \tag{8.10}$$

where $\sigma(z_1 \cdot z_2) = \mathrm{Im}(z_1,z_2)$ and (z_1,z_2) is the positive definite inner product in h^c.[2)] It follows then from (6.9) that $e(z) \to e^{iB(z)}$ provides a *-representation of the Weyl algebra, which is called the free Fock representation of the Weyl algebra, and is given by the state

$$\omega_o(e(z)) = (\Omega_o,\ e^{iB_o(z)}\,\Omega_o) = e^{-\frac{1}{2}\|z\|^2} \ . \tag{8.11}$$

A quasi-free state on the Weyl algebra is a state given by

$$\omega_s(e(z)) = e^{-\frac{1}{2}s(z,z)} \ , \tag{8.12}$$

where s is a real symmetric and positive definite form on $h^c = h \oplus h$ as a real Hilbert space. For a discussion of the quasi-free states see Ref.[34],2) and Ref. [34],3). It follows from (8.10) that $e(0) = 1$ and therefore also that $e(-z) = e(z)^{-1}$, which is equal to $e(z)^*$. Hence in any *-representation of the Weyl algebra $e(z)$ is represented by a unitary operator. Since $\omega_s(e(z)) = \omega_s(e(-z)) = \omega_s(e(z)^*)$ we have that any state of the form (8.12) gives a *-representation and therefore $e(z)$ is represented in the form $e^{iB_s(z)}$ where the $B_s(z)$ are selfadjoint and satisfy the commutation relations (8.9). It follows then that (8.12) is equivalent with

$$\omega_s(B_s(z_1)B_s(z_2)) = s(z_1,z_2) + i\sigma(z_1,z_2) \ . \tag{8.13}$$

From the fact that

$$\omega_s([B_s(z_1)+iB_s(z_2)][B_s(z_1)-iB_s(z_2)]) \geq 0 \tag{8.14}$$

we get that

$$|\sigma(z_1,z_2)| \leq s(z_1,z_1)^{\frac{1}{2}}\, s(z_2,z_2)^{\frac{1}{2}} , \tag{8.15}$$

which must be satisfied in order for ω_s to be a positive state on the Weyl algebra. On the other hand, if (8.15) is satisfied then ω_s defines a positive state on the Weyl algebra, and these states are the quasi-free states.

In this section we shall be concerned with the quasi-free states for the Weyl algebra of the harmonic oscillator which are time invariant, and for this reason we shall first define the Hamiltonian (8.2).

Since A is self adjoint we have that e^{itA} is a strongly continuous unitary group on h^c , and $\alpha_t(e(z)) = e(e^{itA}z)$ then gives a one parameter group of *-automorphisms of the Weyl algebra. Since $\omega_o(e(z)) = e^{-\frac{1}{2}\|z\|^2}$ is obviously invariant under α_t , we get that α_t induces a strongly continuous unitary group e^{itH_o} in the representation given by ω_o . Hence Ω_o is an eigenvector with eigenvalue zero for H_o , and one finds easely that H_o is a positive self adjoint operator. This is then the usual definition of the Hamiltonian H_o for the harmonic oscillator, in the free Fock representation.

Since e^{itH_o} is induced by a group of *-automorphisms α_t of the Weyl algebra leaving ω_o invariant, we may consider α_t as the group of time isomorphisms of the Weyl algebra for the harmonic oscillator. Any state of the Weyl algebra invariant under α_t will give a representation in which α_t is unitarily induced and therefore such a representation will also carry a representative for the energy of the harmonic oscillator, i.e. a Hamiltonian. We shall therefore be interested in characterizing the quasi-free states invariant under α_t .

Let us first assume that A is cyclic in h , i.e. there exists a vector $\varphi_o \in h$ which is cyclic for A , so that $P(A)\varphi_o$ is dense in h , where $P(A)$ is an arbitrary polynomial in A . If A is not cyclic in h , then h decompose into a direct sum of closed invariant subspaces each of which is cyclic.

Since φ_o is cyclic in h , it is also cyclic in h^c and by the spectral representation theorem we have that h^c is isomorhic with $L_2(SpA,d\nu) \equiv L_2(d\nu)$, where $d\nu$ is the spectral measure of A given by the cyclic vector φ_o i.e., for any continuous complex function $f(\omega)$ defined on SpA ,

$$(\varphi_o, f(A)\varphi_o) = \int_{SpA} f(\omega)d\nu(\omega) \ . \tag{8.16}$$

This isomorphism is given by

$$f(A)\varphi_o \longleftrightarrow f(\omega) \tag{8.17}$$

for any $f \in C(SpA)$. By (8.16) and the fact that φ_o is cyclic, (8.17) extends by continuity to an isomorphism between h^c and $L_2(d\nu)$. It follows now from (8.17) that, since φ_o belongs to h and is cyclic in h , the Hilbert space h is mapped onto $L_2^R(d\nu)$ i.e. the real subspace consisting of real functions. From (8.17) we get

$$Af(A)\varphi_o \longleftrightarrow \omega f(\omega) \ ,$$

hence we may take $h = L_2^R(d\nu)$, A to be the multiplication by ω on $L_2^R(d\nu)$ and $h^c = L_2(d\nu)$.

A quasi-free state which is given by (8.12) is invariant under α_t if and only if the form $s(z,z)$ is invariant under the transformation $z \to e^{itA}z$, since

$$\alpha_t(e(z)) = e(e^{itA}z) .$$

We recall that $s(z,z)$ is a symmetric form on the real symplectic space $S = h^c$ with the symplectic structure $\sigma(z_1,z_2) = \operatorname{Im}(z_1,z_2)$, where (z_1,z_2) is the inner product in the complex Hilbert space h^c , which satisfies the condition (8.15). Since e^{itA} is unitary on h^c, it leaves σ invariant and is therefore a symplectic transformation of S and so induces a *-automorphism α_t of the Weyl algebra.

That ω_s , given by (8.12), is invariant under α_t , is obviously equivalent with the positive symmetric form $s(z_1,z_2)$ defined on S being invariant under the symplectic transformation $z \to e^{itA}z$. Let us recall that $s(z_1,z_2)$ is symmetric and bilinear only on the real space $S = h^c$, i.e. bilinear only under real linear combinations.

Let now f_1 and f_2 be Fourier transforms of real bounded signed measures μ_1 and μ_2

$$f_i(\omega) = \int e^{i\omega t}\, d\mu_i(t) \tag{8.18}$$

It follows from the spectral representation theorem that

$$f_i(A) = \int e^{itA}\, d\mu_i(t) . \tag{8.19}$$

Hence

$$s(f_1(A)\varphi_o,f_2(A)\varphi_o) = \iint s(e^{it_1A}\varphi_o,\ e^{it_2A}\varphi_o)d\mu_1(t_1)d\mu_2(t) .$$

By the invariance of s under $z \to e^{itA}z$ we then get

$$\begin{aligned} s(f_1(A)\varphi_o,f_2(A)\varphi_o) &= \iint s(\varphi_o,\ e^{i(t_2-t_1)A}\varphi_o)d\mu_1(t_1)d\mu_2(t_2) \\ &= \iint s(\varphi_o,\ e^{itA}\varphi_o)d\mu_1(t_1)d\mu_2(t+t_1) . \end{aligned} \tag{8.20}$$

Now

$$\bar{f}_1 f_2(\omega) = \overline{f_1}(\omega)\cdot f_2(\omega) = \iint e^{i\omega(t_2-t_1)}\, d\mu_1(t_1)d\mu_2(t_2) \qquad (8.21)$$
$$= \iint e^{i\omega t}\, d\mu_1(t_1)d\mu_2(t+t_1)\ ,$$

from which we get that

$$s(\varphi_o, \overline{f_1}f_2(A)\varphi_o) = \iint s(\varphi_o,\ e^{itA}\,\varphi_o)d\mu(t_1)d\mu_2(t+t_1)\ .$$

Hence we have proved

$$s(f_1(A)\varphi_o, f_2(A)\varphi_o) = s(\varphi_o, \overline{f_1}f_2(A)\varphi_o) \qquad (8.22)$$

for any f_1 and f_2 which are Fourier transforms of bounded signed real measures. Hence (8.22) holds for all f_1 and f_2 in $\mathcal{S}(R)$ such that $f_i(\omega) = \overline{f_i}(-\omega)$. Now we obviously have that the functions f in $\mathcal{S}(R)$ satisfying $f(\omega) = \bar{f}(-\omega)$ are dense in $C_o[0,\infty]$, the space of continuous functions on $[0,\infty]$ tending to zero at infinity. Hence by continuity, since $Sp(A) \subset [0,\infty]$, (8.22) holds for all continuous bounded functions tending to zero at infinity. The strong continuity of $s(z,z)$ follows from the fact that $s(z_1,z_2)$ is bilinear and $s(z,z)$ is positive and defined for all z , and this gives that $s(f_1(A)\varphi_o, f_2(A)\varphi_o)$ is continuous in the strong $L_2(d\nu)$ topology. We have thus that

$$s(\varphi_o, \overline{f_1}f_2(A)\varphi_o) = s(f_1(A)\varphi_o, f_2(A)\varphi_o)\ , \qquad (8.23)$$

for continuous f_1 and f_2 being zero at infinity, and in fact by the strong $L_2(d\nu)$ continuity also for all f_1 and f_2 in $L_2(d\nu)$. From this we get that, if $g \geq 0$, then

$$s(\varphi_o, g(A)\varphi_o) = s(g^{\frac{1}{2}}(A)\varphi_o, g^{\frac{1}{2}}(A)\varphi_o) \geq 0\ , \qquad (8.24)$$

so that $s(\varphi_o, g(A)\varphi_o)$ defines a bounded positive linear functional on the space of continuous functions, hence a measure which is obviously absolutely continuous with respect to the spectral measure. So we have proved that

$$s(f_1(A)\varphi_o, f_2(A)\varphi_o) = \int \bar{f}_1 f_2(\omega)\rho(\omega)d\nu(\omega) \ , \qquad (8.25)$$

where ρ is a positive measurable function, and the right hand side is also the representation of s in the spectral representation of h^c. The condition (8.15) for the positivity of the state ω_s is obviously equivalent to the condition

$$\rho(\omega) \geq 1 \quad \text{a.e.}$$

We should remark that we only assumed $s(z_1, z_2)$ to be bilinear under real linear combinations, but in fact the invariance of s under $z \to e^{itA}z$ gives that s is of the form (8.25), which is actually a sesquilinear form. By the fact that $s(z_1, z_2)$ is everywhere defined, we get that $\rho(\omega)$ is bounded almost everywhere. So in fact we may write (8.25) also as

$$s(z_1, z_2) = (z_1, Bz_2) \ ,$$

where B is a bounded symmetric operator commuting with A, such that $B \geq 1$.

Theorem 8.1

Let h be a real separable Hilbertspace and A be a positive self adjoint operator on h such that zero is not an eigenvalue of A. Let $S = h^c$ be the real symplectic space with symplectic structure given by $\sigma(z_1, z_2) = \operatorname{Im}(z_1, z_2)$, where (z_1, z_2) is the inner product on the complex Hilbert space h^c. $z \to e^{itA}z$ is then a group of symplectic transformations on S and generates therefore a group α_t of *-automorphisms of the Weyl algebra over S, where the Weyl algebra is the algebra generated by $e(z)$, $z \in S$ with the multiplication

$$e(z_1)e(z_2) = e^{-i\sigma(z_1,z_2)}\, e(z_1+z_2)$$

and *-operation given by $e(z)^* = e(-z)$. A quasi-free state of the Weyl algebra is a state of the form

$$\omega_s(e(z)) = e^{-\frac{1}{2}s(z,z)} ,$$

where $s(z,z)$ is a positive bilinear form on the real space S.

A necessary and sufficient condition for ω_s to be a quasi-free state invariant under α_t, is that there exists a bounded, symmetric operator C on the complex Hilbert space h^c such that $C \geq 1$, C commutes with A and

$$s(z_1,z_2) = (z_1,Cz_2) ,$$

where $(,)$ is the inner product in the complex Hilbert space h^c.

<u>Proof</u>. If A is cyclic in h then the theorem is already proved. If A is not cyclic in h, h decomposes in a direct sum $h = \bigoplus_i h_i$, and, in each component h_i, A is cyclic, i runs over at most a countable index set since h is separable. Let φ_i be a cyclic vector in h_i. Then, with f_i continuous, $\sum_{i=1}^{n} f_i(A)\varphi_i$ is dense in h, and in the same way as in the cyclic case we prove that

$$s\Big(\sum_{i=1}^{n} f_i(A)\varphi_i, \sum_i g_i(A)\varphi_i\Big) = \sum_{ij} s(\varphi_i, \overline{f_i}(A)f_j(A)\varphi_j)$$

$$= \int \bar{f}_i(\omega)\rho_{ij}(\omega)f_j(\omega)d\nu(\omega) .$$

The last line is actually the spectral resolution $\rho_{ij}(\omega)$ of a operator C that commutes with A. This proves the theorem. $\square$

For simplicity of notation we shall now assume that A acts cyclic in h. This is in reality no restriction since, if not, then h decomposes, $h = \oplus_n h_n$, in at most a countable sum of cyclic subspaces.

Let now $\mathcal{H}$ be the real Hilbert space of h valued functions on E which are continuous and such that the norm $|\gamma|$ is finite, where

$$|\gamma|^2 = \int_{-\infty}^{\infty}(\dot{\gamma}\cdot\dot{\gamma}+ \gamma A^2\gamma)d\tau$$

and $\gamma\cdot\gamma$ is the inner product in h. On this Hilbert space the classical action $S(\gamma)$ for the harmonic oscillator

$$S(\gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty} \dot{\gamma}\cdot\dot{\gamma}\, d\tau - \tfrac{1}{2}\int_{-\infty}^{\infty} \gamma(\tau)\cdot A^2\gamma(\tau)d\tau$$

is a bounded quadratic form. Let φ be a cyclic vector for A in h, we know then that h may be identified with $L_2^R(d\nu)$, and therefore $\mathcal{H}$ with the real functions in two real variables with norm

$$|\gamma|^2 = \iint((\tfrac{\partial\gamma}{\partial t})^2 + \omega^2\gamma^2(t,\omega))dt\, d\nu(\omega) , \tag{8.28}$$

and recalling that zero is not an eigenvalue of A, so that the set $\{0\}$ has ν-measure zero, we see that (8.28) defines a Hilbert norm. Introducing now the Fourier transform $\hat{\gamma}(p,\omega)$ of $\gamma(t,\omega)$ with respect to t, we get

$$|\gamma|^2 = \iint|\hat{\gamma}(p,\omega)|^2(p^2+\omega^2)dp\, d\nu(\omega) , \tag{8.29}$$

so that $\gamma \to \hat{\gamma}$ is an isometry of $\mathcal{H}$ onto the real subspace of $L_2^c[(p^2+\omega^2)dp\, d\nu(\omega)]$ consisting of functions satisfying

$$\overline{\hat{\gamma}(p,\omega)} = \hat{\gamma}(-p,\omega) . \tag{8.30}$$

Let now D be the subspace of functions in $\mathcal{H}$ consisting of functions γ such that $\hat{\gamma}(p,\omega)$ is continuous in ω and continuously differentiable in p with norm

$$\|\gamma\| = |\gamma| + \sup_{\omega,p} \left|\frac{\partial\hat{\gamma}}{\partial p}(p,\omega\right| . \tag{8.31}$$

We define a bounded symmetric operator B by

$$(\gamma,B\gamma)=2S(\gamma) , \tag{8.32}$$

with domain $D(B)$ consisting of functions $\gamma(t,\omega)$ which are continuous and with compact support in R^2 . For $\gamma \in D(B)$ we have that

$$\hat{\gamma}(p,t) = \frac{1}{\sqrt{2\pi}} \int e^{ipt} \gamma(t,\omega)dt \tag{8.33}$$

is obviously continuous in ω and p and continuously differentiable in p . Now the Fourier transform of $B\gamma$ is, by (8.32), given by

$$\widehat{B\gamma}(p,\omega) = \frac{p^2-\omega^2}{p^2+\omega^2} \hat{\gamma}(p,\omega) . \tag{8.34}$$

From this it follows that the range $R(B)$ of B consists of functions, the Fourier transform of which are continuously differentiable in p with uniformly bounded derivatives and continuous in ω . Hence we have

$$R(B) \subset D . \tag{8.35}$$

Let now $C \geq 0$ be a bounded symmetric operator on h commuting with A . Since A acts cyclically in h, we have that C is represented by a bounded measurable function $c(\omega) \geq 0$ a.e.
We now define the symmetric and continuous form $\Delta_C(\gamma_1,\gamma_2)$ on $D \times D$ by

$$\Delta_C(\gamma_1,\gamma_2) = \int d\nu(\omega)\Big\{P\int_R \overline{\hat{\gamma}_1(p,\omega)}\frac{(p^2+\omega^2)^2}{p^2-\omega^2}\hat{\gamma}_2(p,\omega)dp$$
$$- i\pi\, c(\omega)\int_R \overline{\hat{\gamma}_1(p,\omega)}\,\delta\,(p^2-\omega^2)(p^2+\omega^2)^2\,\hat{\gamma}_2\,(p,\omega)dp\Big\} . \tag{8.36}$$

From (8.34) we have, for $\gamma_1 \in D$ and $\gamma_2 \in D(B)$,

$$\Delta_C(\gamma_1,B\gamma_2) = (\gamma_1,\gamma_2) . \tag{8.37}$$

Since $c(\omega) \geq 0$ we also have that Δ_C has non positive imaginary part, so that $\mathcal{H}$, D, B and Δ_C satisfy the conditions of Definition 4.0 for the integral on $\mathcal{H}$ normalized with respect to Δ_C .

Let now $u \in h$, we define the element $\gamma_u^s(t) \in \mathcal{H}$ by

$$(\gamma_u^s,\gamma) = u\cdot\gamma(s) \tag{8.38}$$

and we have then that

$$\gamma_u^s(t) = \frac{1}{2A}\, e^{-|t-s|A}\cdot u , \tag{8.39}$$

so that

$$(\gamma_u^s,\gamma_u^s) = \tfrac{1}{2}u\,.\,A^{-1}u , \tag{8.40}$$

which implies that $\gamma_u^s \in \mathcal{H}$ if $u \in D(A^{-\frac{1}{2}})$. Furthermore we get by computation that

$$\hat{\gamma}_u^s(p,\omega) = \frac{1}{\sqrt{2\pi}}\,\frac{e^{ips}}{p^2+\omega^2}\cdot u(\omega) , \tag{8.41}$$

Hence, for $u(\omega)$ continuous and bounded and in $D(A^{-\frac{1}{2}})$, that $\hat{\gamma}_u^s(p,\omega)$ is in D . Moreover some futher computations give

$$\Delta_C(\gamma_u^s,\gamma_v^t) = -\,u\cdot\Big[\frac{1}{2A}\sin|t-s|A+\frac{i}{2A}C\cos(t-s)A\Big]v \;. \tag{8.42}$$

Let now $G_C(s-t)$ be the self adjoint operator in h defined by

$$G_C(s-t) = -\frac{1}{2A}\,[\sin|t-s|A + i\,C\cos(t-s)A]\ . \qquad (8.43)$$

Then

$$\Delta_C(\gamma_u^s, \gamma_v^t) = u\cdot G_C(s-t)v\ . \qquad (8.44)$$

Let $u_1, \ldots, u_n$ be in $D(A^{-\frac{1}{2}})$ and such that $u_1(\omega), \ldots, u_n(\omega)$ are continuous and bounded. Then $\gamma_{u_i}^{t_i} \in D$ for $i = 1, \ldots, n$. Hence, with

$$f(\gamma) = e^{i\sum_{j=1}^{n} u_j\cdot\gamma(t_j)} = e^{i\sum_{j=1}^{n}(\gamma, \gamma_{u_j}^{t_j})}\ ,$$

we get $f(\gamma) \in \mathcal{F}(D^*)$, so that we may compute

$$\int_{\mathcal{H}}^{\Delta_C} e^{iS(\gamma)}\, e^{i\sum_{j=1}^{n} u_j\,\gamma(t_j)}\, d\gamma = e^{-\frac{i}{2}\sum_{jk} u_j G_C(t_j-t_k)u_k}\ . \qquad (8.45)$$

Let us now consider the quasi-free state of theorem 8.1,

$$\omega_C(e(z)) = e^{-\frac{1}{2}(z,Cz)}\ , \qquad (8.46)$$

where $C \geq 1$ and commutes with A. By (8.4) we have that, for $u \in D(A^{-\frac{1}{2}})$,

$$\varphi(u) = a^*((2A)^{-\frac{1}{2}}u) + a((2A)^{-\frac{1}{2}}u) \qquad (8.47)$$

is the quantization of the linear function $u\cdot x$ defined on h. In conformity with the notation used in section 7 we define

$$u\cdot x(t) = \alpha_t(\varphi(u))\ , \qquad (8.48)$$

where α_t is the group of time automorphisms given by (8.20). In fact we have then that, expressed in the Weyl algebra for the harmonic oscillator,

$$u\cdot x(t) = e(e^{itA}(2A)^{-\frac{1}{2}}u) \ . \tag{8.49}$$

Let now $u_1,\ldots,u_n$ be in $D(A^{-\frac{1}{2}})$ and consider, for $t_1\leq\ldots\leq t_n$,

$$\omega_C(e^{iu_1\cdot x(t_1)} \ldots e^{iu_n\cdot x(t_n)}) \ . \tag{8.50}$$

We get easely, using (8.10) and (8.46), that (8.50) is equal to

$$\omega_C(e^{iu_1\cdot x(t_1)} \ldots e^{iu_n\cdot x(t_n)}) = e^{-\frac{i}{2}\sum_{jk} u_j\cdot G_C(t_j-t_k)u_k} \ . \tag{8.51}$$

So by (8.45) we have proved

$$\omega_C(e^{iu_1\cdot x(t_1)} \ldots e^{iu_n\cdot x(t_n)}) = \int_{\mathcal{H}}^{\Delta} e^{iS(\gamma)} e^{i\sum_{j=1}^{n} u_j\cdot\gamma(t_j)} d\gamma \ . \tag{8.52}$$

We state this fact in the following theorem:

<u>Theorem 8.2</u>

Let h be a real separable Hilbert space and A a positive self adjoint operator on h such that zero is not an eigenvalue of A . The classical action for the harmonic oscillator on h is given by

$$S(\gamma) = \frac{1}{2}\int_{-\infty}^{\infty} (\dot{\gamma}\cdot\dot{\gamma} - \gamma\cdot A^2\gamma)dt \ .$$

Let α_t be the time automorphism of the Weyl algebra for the corresponding quantum system. Let $C \geq 0$ be a bounded self adjoint operator commuting with A , then the Fresnel integral relative to $2S(\gamma)$, normalized with respect to Δ_C , whe Δ_C is given in (8.36), exists and for $C \geq 1$ this Fresnel integral induces a quasi-free state on the Weyl algebra, invariant under α_t, by the formula

$$\omega_C(e^{iu_1\cdot x(t_1)}\cdots e^{iu_n\cdot x(t_n)}) = \int\limits_{\mathcal{H}}^{\Delta_C} e^{iS(\gamma)}\, e^{i\sum_{j=1}^{n} u_j\gamma(t_j)}\, d\gamma\ ,$$

with $u_1,\ldots,u_n$ in $D(A^{-\frac{1}{2}})$ and $t_1\leq\ldots\leq t_n$.

Moreover any invariant quasi-free state on the Weyl algebra is obtained in this way. In particular, if $C = 1$ we get the free Fock state, and if $C = \mathrm{cotgh}(\frac{\beta}{2}A)$ we get the free Gibbs-state at temperature $1/\beta$.

Proof: The first part is already proved. The moreover part follows from theorem 8.1. That we get the free Fock state for $C = 1$ follows by direct inspection and that we get the free Gibbs-state with $C = \mathrm{cotgh}(\frac{\beta}{2}A)$ follows from the form of $G_C(s-t)$ given by (8.43) and Ref. [33], 3) formula (3.32). This proves the theorem.

$\square$

Remark. We have thus, in particular, that the Fresnel integrals relative to the quadratic form $2S(\gamma)$ on $\mathcal{H}$ correspond, in the sense of theorem 8.2, to the linear functionals on the Weyl algebra given by

$$\omega_C(e(z)) = e^{-\frac{1}{2}(z,Cz)}$$

where $C \geq 0$ and commutes with A . However these functionals are positive states on the Weyl algebra only in the case $C \geq 1$.

9. The Feynman history integrals for the relativistic quantum boson field.

The free relativistic scalar boson field in n space dimensions is a harmonic oscillator in the sense of the previous section, with $h = L_2^R(R^n)$ and $A^2 = -\Delta + m^2$, where Δ is the Laplacian as a self adjoint operator on $L_2^R(R^n)$ and m is a non-negative constant called the mass of the field. Because of the importance of this physical system we shall give it a more detailed treatment.

We shall first discuss the free relativistic boson field i.e. the system with a classical action given by

$$S(\varphi) = \tfrac{1}{2} \iint_{R^n} [(\tfrac{\partial\varphi}{\partial t})^2 - \sum_{i=1}^{n} (\tfrac{\partial\varphi}{\partial x_i})^2 - m^2\varphi^2] d\vec{x}\, dt \;. \tag{9.1}$$

Let $\mathcal{H} = \mathcal{H}_1$ be a real Sobolev space, namely the Hilbert space of real valued functions φ over R^{n+1} for which the norm $|\varphi|$ is finite:

$$|\varphi|^2 = \int_{R^{n+1}} [(\tfrac{\partial\varphi}{\partial t})^2 + \sum_{i=1}^{n} (\tfrac{\partial\varphi}{\partial x_i})^2 + \varphi^2] d\vec{x}\, dt \;, \tag{9.2}$$

Then $S(\varphi)$ is a bounded continuous quadratic form on $\mathcal{H}$ and we define a bounded symmetric operator B on $\mathcal{H}$ by

$$(\varphi, B\varphi) = 2S(\varphi) \;, \tag{9.3}$$

with domain $D(B)$ equal to the set of functions in $\mathcal{H}$ with compact support in R^{n+1}. The Fourier transformation $\varphi \to \hat{\varphi}$

$$\hat{\varphi}(p) = (2\pi)^{-\frac{n+1}{2}} \int e^{ipx} \varphi(x) dx \;, \tag{9.4}$$

with $x = \{t, \vec{x}\}$, is an isomorphism of $\mathcal{H}$ with the real subspace of $L_2((p^2+m^2)dp)$ consisting of functions $\hat{\varphi}$ such that

$$\bar{\hat{\varphi}}(p) = \hat{\varphi}(-p) \;. \tag{9.5}$$

Let D be the linear subspace of functions φ in $\mathcal{H}$ whose Fourier transforms $\hat{\varphi}(p)$ are continuously differentiable with bounded derivatives. The norm in D is given by

$$\|\varphi\| = |\varphi| + \sup_{i,p} \left|\frac{\partial\hat{\varphi}}{\partial p_i}\right| . \tag{9.6}$$

If $\varphi \in D(B)$ then $\hat{\varphi}$ is a smooth function and the Fourier transform of $B\varphi$ is given by

$$\widehat{B\varphi}(p) = \frac{p_o{}^2 - \vec{p}^2 - m^2}{p^2+1}\,\hat{\varphi}(p) , \tag{9.7}$$

with $p = \{p_o, \vec{p}\}$.

Now, since $\varphi \in D(B)$, we have that

$$\frac{\partial\hat{\varphi}}{\partial p_j} = \int_C e^{ipx}(ix_j)\varphi(x)dx , \tag{9.8}$$

where C is compact. Since φ is in $L_2(R^{n+1})$ and $C = \operatorname{supp}\varphi$ is compact, φ is also in L_1 , so that $\frac{\partial\hat{\varphi}}{\partial p_j}$ is bounded and continuous. This gives immediately, from (9.7), that $B\varphi$ is in D . Let $c(\vec{p})$ be a measurable non negative function on R^n . We then define a continuous and bounded symmetric bilinear form $\Delta_C(\varphi,\psi)$ on $D\times D$ by

$$\begin{aligned} \Delta_C(\varphi,\psi) &= P\int_{R^{n+1}} \bar{\hat{\varphi}}(p)\,\frac{(p^2+1)^2}{p_o^2-\vec{p}^2-m^2}\,\hat{\psi}(p)dp \\ &- i\pi\int c(\vec{p})\bar{\hat{\varphi}}(p)\delta(p_o^2-\vec{p}^2-m^2)(p^2+1)^2\hat{\psi}(p)dp , \end{aligned} \tag{9.9}$$

where $P\int \frac{f(p)}{p_o^2-\vec{p}^2-m^2}\,dp = \int_{R^n}[P\int_R \frac{f(p_o,\vec{p})}{p_o^2-\vec{p}^2-m^2}\,dp_o]d\vec{p}$

for any smooth function $f(p)$ and $P\int_R \frac{f(p_o\vec{p})}{p_o^2-\vec{p}^2-m^2}\,dp_o$ is the principal value integral. Since the first term in (9.9) is real, we see that

$$\operatorname{Im} \Delta_C(\varphi,\varphi) \leq 0 \ .$$

Let now $\varphi \in D$ and $\psi \in D(B)$. Then we get from (9.9) that

$$\Delta_C(\varphi,B\psi) = (\varphi,\psi) \ , \tag{9.10}$$

and hence we have verified that $\mathcal{H}$, D, B and Δ_C satisfy the conditions of definition 4.0 for the Fresnel integral with respect to the classical action $S(\varphi)$ and normalized according to Δ_C . This Fresnel integral will also be called Feynman history integral, and if we want to emphasize the dependence on the non negative function c we shall call it the Feynman history integral relative to C .

Let now h^c be the complexification of $h = L_2^R(R^n)$, and consider the Weyl algebra over h^c . The quantized field $\Phi(\vec{x})$ at time zero is then given in terms of the Weyl algebra. In fact, for any $f \in h$ such that $f \in D(A^{-\frac{1}{2}})$, we have

$$e^{i\Phi(f)} = e((2A)^{-\frac{1}{2}}f) \ , \tag{9.11}$$

where $\Phi(f) = \int\Phi(\vec{x})f(\vec{x})d\vec{x}$ and $A = \sqrt{-\Delta+m^2}$. For the definition of the Weyl algebra over h^c see the previous section. The time automorphism of the Weyl algebra was given by

$$\alpha_t^o(e(g)) = e(e^{itA}g) \ . \tag{9.12}$$

Now $h^c = L_2(R^n)$ carries a natural unitary representation of the translation group R^n , so that, for any $a \in R^n$, $g \to g_a$ with $g_a(x) = g(x-a)$ is a unitary transformation of h^c . Since it is unitary it is also symplectic, hence

$$\beta_a(e(g)) = e(g_a) \tag{9.13}$$

is a *-automorphism of the Weyl algebra. We have the following theorem:

Theorem 9.1

Let $h = L_2^R(R^n)$ and $h^c = L_2(R^n)$. Let $g \in h^c$ and $e(g)$ the corresponding element in the Weyl algebra over h^c. The quantized time zero field $\Phi(f)$ is then expressed in terms of this Weyl algebra by

$$e^{i\Phi(f)} = e\,((2A)^{-\frac{1}{2}}f)$$

for any $f \in h$.

Any quasi-free state which is invariant under the time automorphisms α_t^o and also under the space automorphisms β_a is of the form

$$\omega_C(e(g)) = e^{-\frac{1}{2}(g,Cg)} ,$$

where

$$(g,Cg) = \int_{R^n} c(\vec{p})|\hat{g}(p)|^2 d\vec{p}$$

and $c(\vec{p})$ is a bounded measurable function such that

$$c(\vec{p}) \geq 1 .$$

Proof:

That $\omega_C(e(g)) = e^{-\frac{1}{2}(g,Cg)}$, where C is a bounded symmetric operator on h^c such that $C \geq 1$ and C commutes with A, follows from theorem 8.1. Now, since ω_C is to be invariant under β_a, we get

$$\omega_C(e(g)) = \omega_C(\beta_a(e(g))) = \omega_C(e(g_a)) ,$$

hence

$$(g,Cg) = (g_a, Cg_a) ,$$

so that C is an operator in $L_2(R^n)$ which commutes with translation. Hence C is of the form given in the theorem. This proves the theorem. $\square$

Let now

$$e^{i\Phi_t(f)} = \alpha_t^o(e^{i\Phi(f)}) . \tag{9.14}$$

We have the following:

Theorem 9.2

Let $h = L_2^R(R^n)$ and $h^c = L_2(R^n)$. The classical action for the free relativistic scalar boson field in R^n is given by

$$S(\varphi) = \tfrac{1}{2}\int_R \int_{R^n} \left(\left(\frac{\partial\varphi}{\partial t}\right)^2 - \sum_{j=1}^n \left(\frac{\partial\varphi}{\partial x_j}\right)^2 - m^2\varphi^2\right) d\vec{x}\, dt ,$$

where m is a non negative constant called the mass of the free field. Let α_t and β_a be the time and space automorphisms of the Weyl algebra over h^c . Let $c(\vec{p}) \geq 0$ be a non negative bounded measurable function on R^n . Then the Fresnel integral relative to $2S(\varphi)$, normalized with respect to Δ_C , exists, where Δ_C is given in (9.9), and is called the Feynman history integral relative to C . If $c(\vec{p}) \geq 1$, the corresponding Feynman history integral defines a quasi-free state ω_C on the Weyl algebra for the scalar field, i.e. the Weyl algebra over h^c , and ω_C is invariant under the time and space automorphisms α_t and β_a of the Weyl algebra. The correspondence between the history integral and the state is given, for $t_1 \leq \ldots \leq t_n$, by the formula

$$\omega_C(e^{i\Phi_{t_1}(f_1)} \ldots e^{i\Phi_{t_n}(f_n)}) = \int_{\mathcal{H}}^{\Delta_C} e^{iS(\varphi)}\, e^{i\sum_{j=1}^n \int\varphi(\vec{x},t_j) f_j(\vec{x}) d\vec{x}}\, d\varphi ,$$

if $f_1,\dots,f_n$ are in $D(A^{-\frac{1}{2}})$.

Moreover any quasi-free state invariant under spacetime translations is obtained in this way. In particular, for $c(\vec{p}) = 1$ we get the free Fock representation and for $c(\vec{p}) = \operatorname{cotgh}(\frac{\beta}{2}\omega(\vec{p}))$, with $\omega(\vec{p}) = \sqrt{\vec{p}^2+m^2}$, we get the free Gibbs state at temperature $1/\beta$.

Proof: That the Fresnel integral relative to $2S(\varphi)$ normalized with respect to Δ_C exists, with Δ_C given by (9.9), was proven before. Consider now, for $t_1 \leq t_2 \leq\dots\leq t_n$ and $f_1,\dots,f_n$ in $D(A^{-\frac{1}{2}})$, $\omega_C(e^{i\Phi_{t_1}(f_1)}\dots e^{i\Phi_{t_n}(f_n)})$, where ω_C is the invariant quasi-free state of theorem 9.1 and $e^{i\Phi_t(f)}$ is defined by (9.14). We have then, using (9.14) and (9.11):

$$\omega_C(e^{i\Phi_{t_1}(f_1)}\dots e^{i\Phi_{t_n}(f_n)}) = \omega_C(\alpha^o_{t_1}(e^{i\Phi(f_1)})\dots\alpha^o_{t_n}(e^{i\Phi(f_n)}))$$

$$= \omega_C(\alpha^o_{t_1}(e((2A)^{-\frac{1}{2}}f_1)) \dots \alpha^o_{t_n}(e((2A)^{-\frac{1}{2}}f_n))) .$$

Hence, using (9.12):

$$\omega_C(e^{i\Phi_{t_1}(f_1)}\dots e^{i\Phi_{t_n}(f_n)}) = \omega_C(e(e^{it_1A}(2A)^{-\frac{1}{2}}f_1)\dots$$
$$\dots e(e^{it_nA}(2A)^{-\frac{1}{2}}f_n)) ,$$

and therefore, from the property (8.10) of the multiplication in the Weyl algebra and the fact that, by Theorem 9.1,

$$\omega_C(e(g)) = e^{-\frac{1}{2}(g,Cg)} .$$

we have:

$$\omega_C(e^{i\Phi_{t_1}(f_1)}\dots e^{i\Phi_{t_n}(f_n)}) = e^{-\frac{1}{2}\sum_{jk}\int\overline{\hat{f}_j(\vec{p})}\hat{G}_C(t_j-t_k,\vec{p})\hat{f}_k(\vec{p})d\vec{p}} \tag{9.15}$$

where

$$\hat{G}_C(t,\vec{p}) = \frac{-1}{2\sqrt{\vec{p}^2+m^2}}(\sin|t|\sqrt{\vec{p}^2+m^2} + i\, c(\vec{p})\cos|t|\sqrt{\vec{p}^2+m^2}\,) . \tag{9.16}$$

On the other hand we get easely that

$$F(\varphi) = e^{i\sum_{j=1}^{n}\int\varphi(\vec{x},t_j)f_j(\vec{x})d\vec{x}} \tag{9.17}$$

is in $\mathscr{F}(D^*)$ so that we may compute

$$\int_{\mathscr{H}}^{\Delta_C} e^{iS(\varphi)}\, e^{i\sum_{j=1}^{n}\int\varphi(\vec{x},t_j)f_j(\vec{x})d\vec{x}}\, d\varphi . \tag{9.18}$$

Using now the formula (9.9) for Δ_C, and a representation of the form (8.39) for the linear functional $\int\varphi(\vec{x},t(f(\vec{x})d\vec{x}$ defined on $\mathscr{H}$, we get, with

$$(\psi_t(f),\varphi) \equiv \int\varphi(\vec{x},t)f(\vec{x})d\vec{x} , \tag{9.19}$$

that $\psi_t(f)$ is in D, so that $F(\varphi)$ is in (D^*) and

$$\Delta_C(\psi_s(f),\psi_t(g)) = \int \hat{G}_C(s-t,\vec{p})\bar{\hat{f}}(\vec{p})\hat{g}(\vec{p})dp . \tag{9.20}$$

Hence we obtain from (9.20) and (9.15) the identity in the theorem. That any space and time invariant quasi free state is obtained in this way, follows from the previous theorem. That we get the free vacuum or free Fock representation for $c(\vec{p}) = 1$ is standard and that we get the free Gibbs state at temperature $1/\beta$ if $c(\vec{p}) = \mathrm{cotgh}(\frac{\beta}{2}\omega(\vec{p}))$ is proved in Ref. Ch. 3 formula (3.36). This then proves the theorem. □

Let now ψ be a smooth non negative function on R^n so that $\int\psi(\vec{x})d\vec{x} = 1$ and $\psi(\vec{x}) = 0$ for $|\vec{x}| \geq 1$, and let $\psi_\varepsilon(\vec{x}) = \varepsilon^{-n}\psi(\frac{1}{\varepsilon}\vec{x})$. Then we define the ultraviolet cut-off field $\varphi_\varepsilon(\vec{x})$ by

$$\Phi_\epsilon(\vec{x}) = \int \Phi(\vec{x}-\vec{y})\psi_\epsilon(\vec{y})d\vec{y} \tag{9.21}$$

Let now V be a real function of a real variable such that V is the Fourier transform of a bounded measure i.e. $V \in \mathcal{F}(R)$, we then define the space cut-off interaction V^ϵ_Λ, where Λ is a finite subset of R^n, by

$$V^\epsilon_\Lambda = \int_\Lambda V(\Phi_\epsilon(\vec{x}))d\vec{x} \;. \qquad 1) \tag{9.22}$$

Since

$$V(s) = \int e^{is\alpha}\, d\mu(\alpha) \;, \tag{9.23}$$

(9.22) is defined to be the element associated with the Weyl-algebra given by

$$V^\epsilon_\Lambda = \iint_\Lambda e^{i\Phi_\epsilon(\vec{x})}\, d\mu(\alpha)d\vec{x} \tag{9.24}$$

or, by definition (9.11),

$$V^\epsilon_\Lambda = \iint_\Lambda e(\alpha(2A)^{-\frac{1}{2}}\, \psi^{\vec{x}}_\epsilon)d\mu(\alpha)d\vec{x} \;, \tag{9.25}$$

where $A = \sqrt{-\Delta+m^2}$ and $\psi^{\vec{x}}_\epsilon(\vec{y}) = \psi_\epsilon(\vec{y}-\vec{x})$. The integral (9.25) does not necessarily converge in the topology of the Weyl algebra or, for that matter, in the natural C^*-topology of the Weyl algebra. However, in any representation induced by a state invariant under space translation, the representative of $e(\alpha(2A)^{-\frac{1}{2}}\psi_\epsilon^{\vec{x}})$ is strongly continuous in α and $\vec{x}$, since $e(\alpha(2A)^{-\frac{1}{2}}\psi^{\vec{x}}_\epsilon) = e^{i\alpha\Phi_\epsilon(\vec{x})}$ is strongly continuous in α, because $\Phi_\epsilon(\vec{x})$ is self adjoint and one has

$$e^{i\alpha\Phi_\epsilon(\vec{x})} = U_{\vec{x}}\, e^{i\alpha\Phi_\epsilon(\vec{0})}\, U_{-\vec{x}} \;,$$

where $U_{\vec{x}}$ is a strongly continuous representation of R^n, the state being invariant under space translation. Hence, in any representation induced by a state which is space translation invariant, (9.25) exists as a strong Riemann integral and therefore V^{ϵ}_{Λ} is represented there. Let now ρ be a state on the Weyl algebra which is space translation invariant. In the representation given by ρ we then have V^{ϵ}_{Λ} represented, and we shall use the notation V^{ϵ}_{Λ} for its representative. Assume now also that ρ is invariant under the free time isomorphism α^{o}_{t}. Then, in this representation α^{o}_{t} is induced by a unitary group e^{-itH_o}, where H_o is the self adjoint infinitesimal generator for this unitary group. It follows easely from (9.25) that V^{ϵ}_{Λ} is bounded, hence

$$H = H_o + V^{\epsilon}_{\Lambda} \tag{9.26}$$

is a self adjoint operator in the representation space. Let α_t be the automorphism on the bounded operators of the representation space induced by the unitary group e^{-itH}. Let now ρ be any of the space-time invariant quasi free states of theorem 9.1, we then have the following theorem.

Theorem 9.3

Let ω_C be a quasi free state on the Weyl algebra for the free boson field on R^n, invariant under space and time translations. Let $V \in \mathcal{F}(R)$ and define H_o as the self adjoint operator generating α^{o}_{t} in the representation given by ω_C. Let moreover H be the self adjoint operator in the representation space given by

$$H = H_o + \int_{\Lambda} V(\Phi_{\epsilon}(\vec{x}))d\vec{x} ,$$

and let α_t be the automorphism induced by H on the algebra of bounded operators in the representation space. If $F_1,\dots,F_n$ are in $\mathcal{F}(R)$, and $f_1,\dots,f_n$ in $D(A^{-\frac{1}{2}})$ and $t_1 \leq \dots \leq t_n$, then

$$e^{-i\int_{t_1}^{t_m} V(\varphi_\epsilon(\vec{x},t)\,d\vec{x}\,dt} \prod_{j=1}^{n} F_j\left(\int \Phi(\vec{x},t_j) f_j(\vec{x})\,d\vec{x}\right)$$

is in $\mathcal{F}(D^*)$ and

$$\omega_C(\alpha_{t_1}(F_1(\Phi(f_1)))\alpha_{t_2}(F_2(\Phi(f_2))) \ldots \alpha_{t_n}(F_n(\Phi(f_n))))$$

$$= \int_{\mathcal{H}}^{\Delta C} e^{iS(\varphi)}\, e^{-i\int_{t_1}^{t_m} V(\varphi_\epsilon(\vec{x},t)\,d\vec{x}\,dt} \prod_{j=1}^{n} F_j\left(\int \varphi(\vec{x},t_j) f_j(\vec{x})\,d\vec{x}\right) d\varphi \;,$$

where

$$\varphi_\epsilon(\vec{x},t) = \int \varphi(\vec{x}-\vec{y},t)\,\psi_\epsilon(y)\,dy \;.$$

Proof: The proof of this theorem follows in the same way as the proof of theorem 6.2 by series expansion and use of previous results of this section. $\square$

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Footnotes

Section 1

1) A vivid account of the origins of the idea, influenced particularly by remarks of Dirac [2], has been given by Feynman himself in [3].

2) For the physical foundation see the original work of Feynman and the book by Feynman and Hibbs (Ref. [1]). Also e.g. [7].

3) Actually, Hölder continuous of index less than 1/4, see e.g. [8].

4) For the definition by a "sequential limit", in more general situations, see e.g. [11].

5) Besides the topics touched in this brief historical sketch of the mathematical study of Feynman path integrals in non relativistic mechanics there are others we did not mention, either because they concern problems other than those tackled later in this work or because no clear cut mathematical results are available. Let us mention however three more areas in which Feynman path integrals have been discussed and used, at least heuristically.

   a) Questions of the relation between Feynman's quantization and the usual one: see e.g. [10],6), [31], [35].

   b) Feynman's path integrals on functions defined on manifolds other than Euclidean space, in particular for spin particles. Attempts using the sequential limit and analytic continuation approaches have been discussed to some extent, see e.g. [6], 4), [36] and references given therein. For the analytic continuation approach there is available the well developed theory of Wiener integrals on Riemannian manifolds, see e.g. [37].

   c) An important application of Feynman path integrals is in the discussion of the classical limit, where $\hbar \to 0$: see e.g. [1], [5], [38].

6) The Wightman axioms for a local relativistic quantum field theory (see e.g. [40]) have been proved, in particular.

7) We did not mention here other topics which have some relations to Feynman's approach to the quantization of fields, for much the same reason as in the preceding footnote.5) For a discussion of problems in defining Feynman path integrals for spinor fields see e.g. [4], 2),[39] and references given therein. For the problem of the formulation of Feynman path integrals in general relativity see e.g. [39], [4],2),and references given there.

Section 3.

1) The elementary definitions of mathematical scattering theory are e.g. in [42]. For recent work see e.g. [32].

2) This assumption is actually enough for proving the completeness of the wave operators, in the sense that Range W_+ = Range W_-, see [43]. A slightly weaker condition, sufficient for the existence of the wave operators, is [44]

$$\int |V(x)|^2 (1+|x|)^{2-n+\varepsilon}\, dx < \infty , \quad \varepsilon > 0 .$$

See also [11],3).

Section 5.

1) This condition is however not necessary for the mathematics involved, so that all results actually hold also for complex V.

Section 8.

1) See e.g. [47], [25], 2)

2) See e.g. [48].

Section 9.

1) As mentioned in the introduction, models with these interactions and their limit when the space cut-off is removed ($\Lambda \to R^n$) have been studied before, see e.g. [30], and references given there.

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